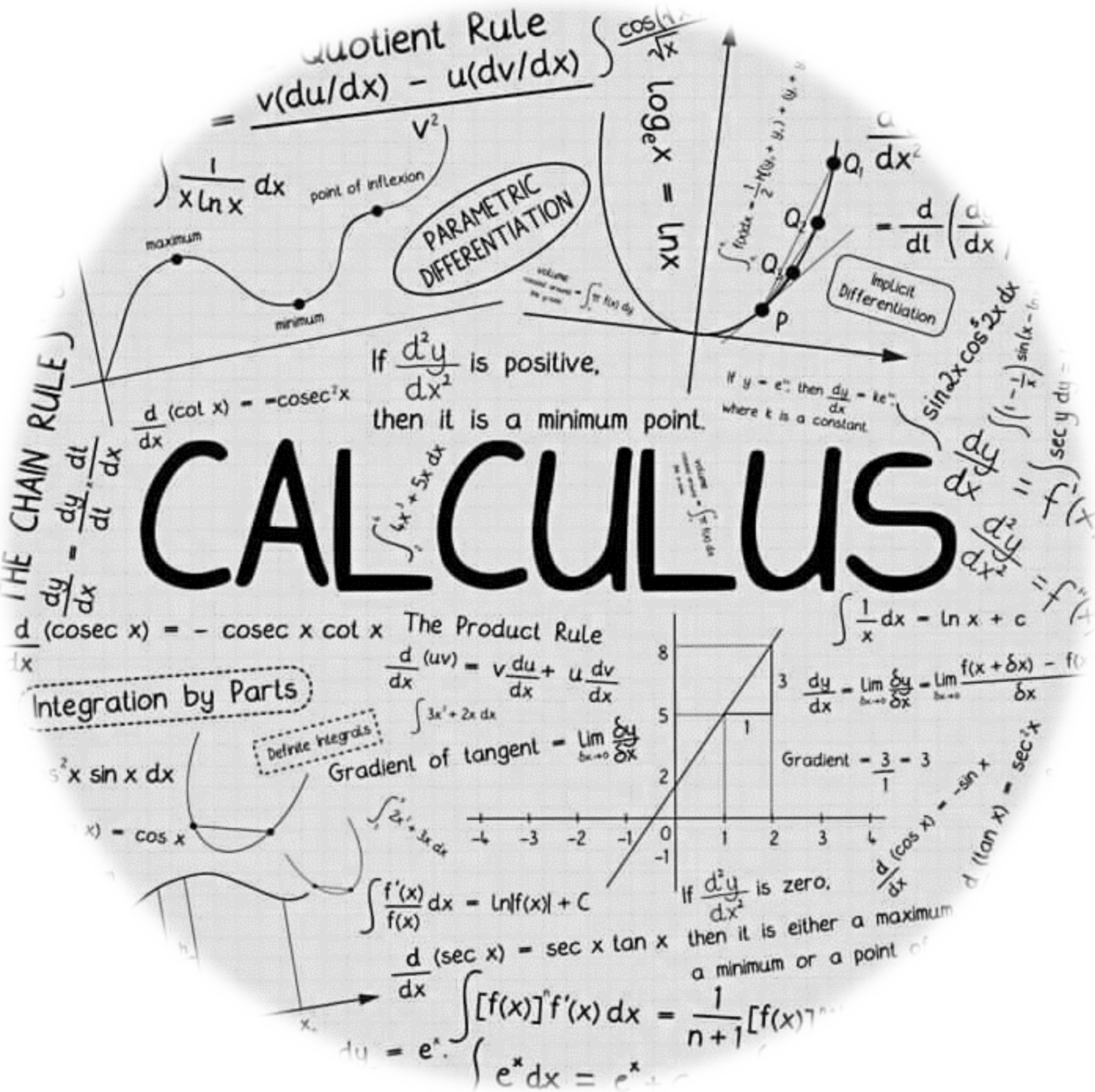
eSpyMath: AP Calculus AB/BC Textbook

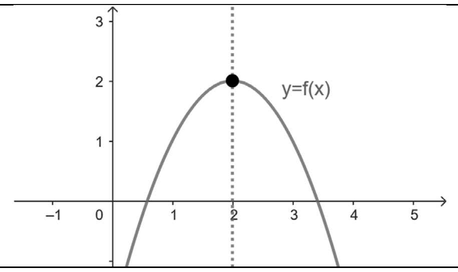
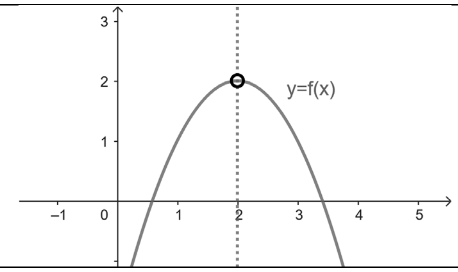
Contents

Chapter 0. Foundations and Previews

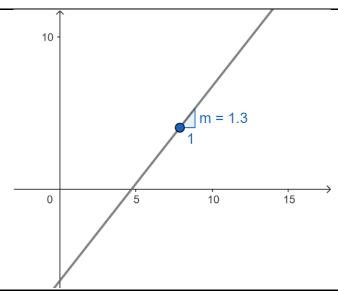
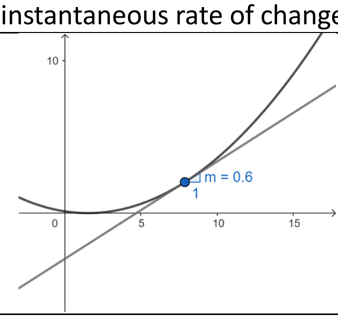


0-1. Why Calculus

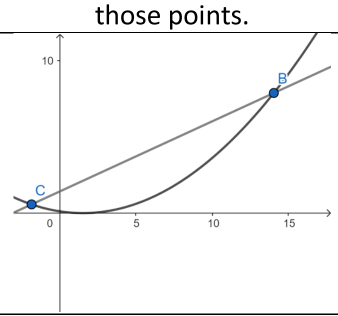
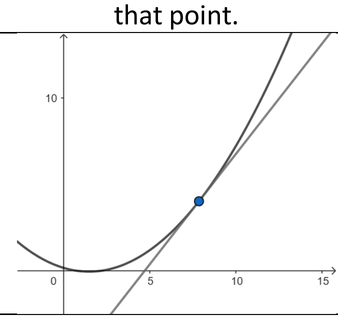
1) Value of $f(x)$ when $x = c$:

Without Calculus	With Differential Calculus
You directly find the value of the function f at $x = c$.	You consider the limit of $f(x)$ as x approaches c, which can be more precise, especially if f is not continuous at c.
	

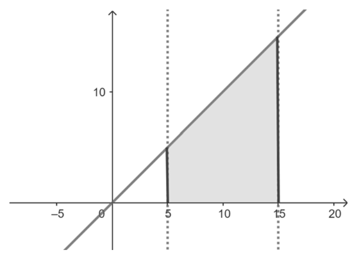
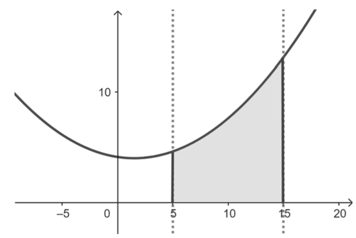
2) Slope of a line:

Without Calculus	With Differential Calculus
The slope is the change in y divided by the change in x ($\Delta y / \Delta x$).	**The slope of a curve at a point** is found using the derivative (dy / dx), representing the instantaneous rate of change.
	

3) Secant line to a curve:

Without Calculus	With Differential Calculus
A secant line intersects the curve at two points, representing the average rate of change between those points.	**A tangent line** touches the curve at one point, representing the instantaneous rate of change at that point.
	

4) Area under the line or curve:

Without Calculus	With Differential Calculus
You find the area by multiplying the length and width of the rectangle/polygon.	**You find the area under a curve** using integration, which can handle more complex shapes.
	

5) Length of a line segment:

Without Calculus	With Differential Calculus
The length is the distance between two points.	You find the **length of an arc** (curved line) using integration.

6) Surface area of a cylinder:

Without Calculus	With Differential Calculus
You calculate the surface area using the formula for a cylinder.	You find **the surface area of a solid** of revolution using integration, which can handle more complex shapes.

7) Mass of a solid of constant density:

Without Calculus	With Differential Calculus
The mass is found by multiplying the volume by the constant density.	You calculate the **mass of a solid** with variable density using integration.

8) Volume of a rectangular solid:

Without Calculus	With Differential Calculus
The volume is found by multiplying length, width, and height.	You find **the volume of a region under a surface** using integration.

0-2. Increasing & Decreasing functions (use in curve sketching)

Examples:

1) Determine the domain and range of the function $g(x)=2e^x$.

The function $g(x)=2e^x$ is a transformation of the basic exponential function e^x.

- Domain: The domain of e^x is all real numbers, $(-\infty,\infty)$. Since $g(x)=2e^x$ is just a vertical stretch, the domain remains the same.
- Domain: $(-\infty,\infty)$
- Range: The range of . e^x . is $(0,\infty)$. Multiplying by 2 stretches the range but does not change its lower or upper bounds.
- Range: $(0,\infty)$

2) Determine the domain and range of the function $g(x)=2\sin(x)$.

The function $g(x)=2\sin(x)$ is a vertical stretch of the basic sine function $\sin(x)$.

- Domain: The domain of $\sin(x)$ is all real numbers, $(-\infty,\infty)$. The vertical stretch does not affect the domain.
- Domain: $(-\infty,\infty)$
- Range: The range of $\sin(x)$ is $[-1,1]$. Multiplying by 2 stretches the range to $[-2,2]$.
- Range: $[-2,2]$

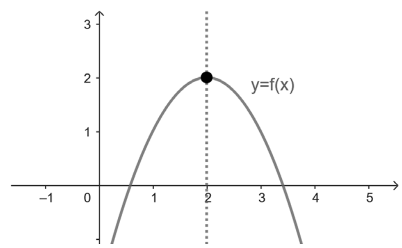
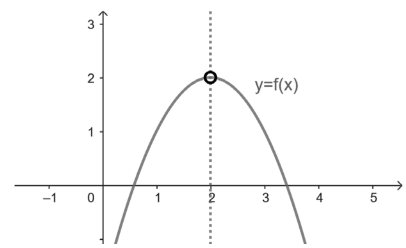
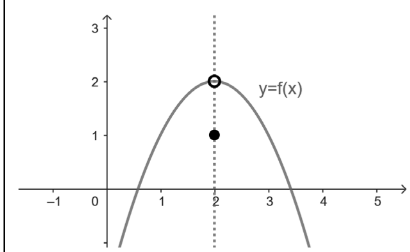
3) Determine the domain and range of the function $g(x) = \dfrac{1}{2x}$**.**

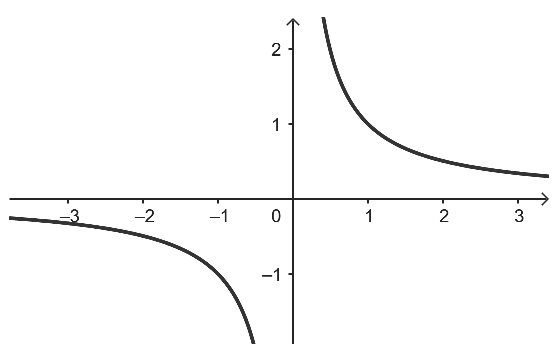
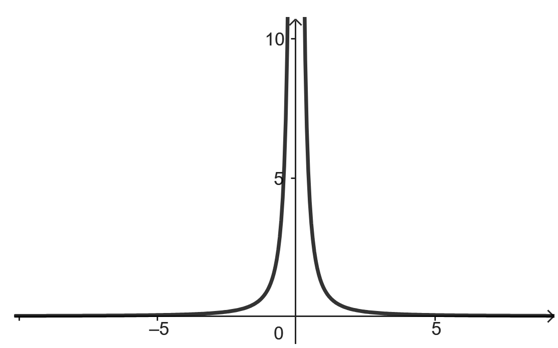
The function $g(x) = \dfrac{1}{2x}$ is a vertical compression of the basic function $\dfrac{1}{x}$.

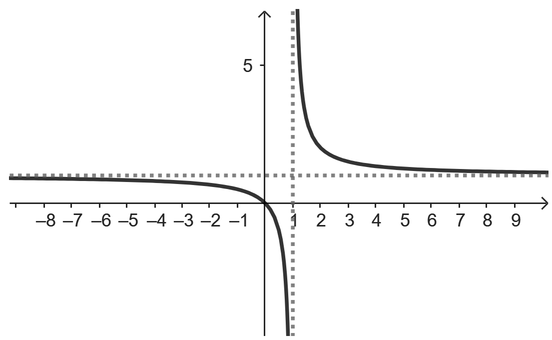
- Domain: The domain of $\dfrac{1}{x}$ is all real numbers except $x = 0$, because division by zero is undefined. The vertical compression does not affect the domain.
- Domain: $(-\infty, 0) \cup (0, \infty)$
- Range: The range of $\dfrac{1}{x}$ is all real numbers except $y = 0$, since $\dfrac{1}{x}$ never equals zero. The vertical compression does not affect the range.
- Range: $(-\infty, 0) \cup (0, \infty)$

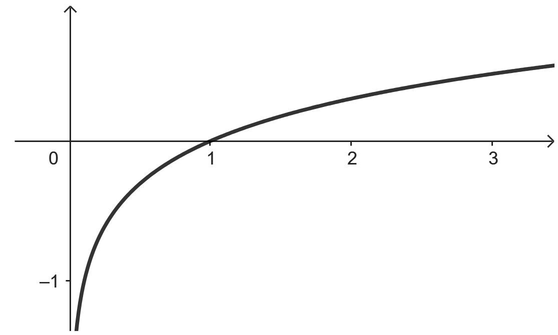
0-3. Limit Foundation

Examples: Find limits for the graph below

		
1) $\lim_{x\to 2} f(x) = 2$	**2)** $\lim_{x\to 2} f(x) = 2$	**3)** $\lim_{x\to 2} f(x) = 2$

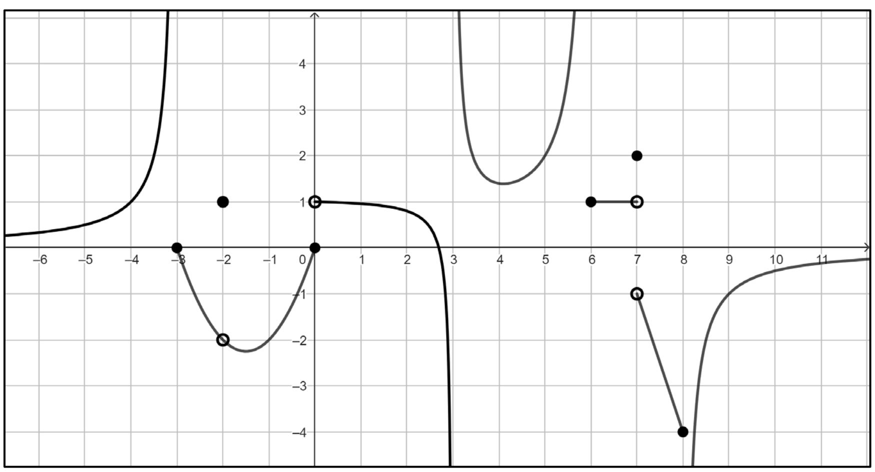
			
4) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$	**5)** $\lim_{x\to 0^+} \frac{1}{x} = \infty$	**6)** $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$	**7)** $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$





8) $\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$	**9)** $\lim_{x\to 1^+} \frac{x}{x-1} = \infty$	**10)** $\lim_{x\to \infty} \log(x) = 0$	**11)** $\lim_{x\to 0^+} \log(x) = -\infty$

12) Given the graph of $f(x)$ above, find the limit.



$\lim_{x\to -\infty} f(x) = 0$	$\lim_{x\to \infty} f(x) = 0$	$\lim_{x\to -3^+} f(x) = 0$	$\lim_{x\to -3^-} f(x) = \infty$
$\lim_{x\to -2^-} f(x) = -2$	$\lim_{x\to -2^+} f(x) = -2$	$\lim_{x\to 0^+} f(x) = 1$	$\lim_{x\to 0^-} f(x) = 0$
$\lim_{x\to 3^+} f(x) = \infty$	$\lim_{x\to 3^-} f(x) = -\infty$	$\lim_{x\to 6^+} f(x) = 1$	$\lim_{x\to 6^-} f(x) = -\infty$
$\lim_{x\to 7^+} f(x) = -1$	$\lim_{x\to 7^-} f(x) = 1$	$\lim_{x\to 8^+} f(x) = -\infty$	$\lim_{x\to 8^-} f(x) = -4$
$f(-2) = 1$	$f(7) = 2$		

0-4. Basic Theorems in Calculus (Preview)

Theorem: Intermediate Value Theorem, IVT

If **f is continuous** on a closed interval [a, b] and k is any number between f(a) and f(b), then there exists at least one number c in (a, b) such that f(c) = k.

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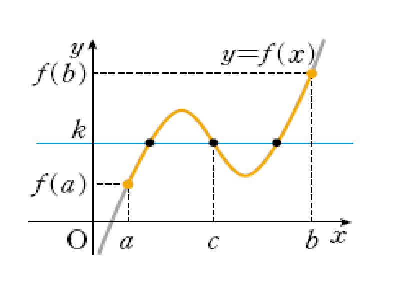
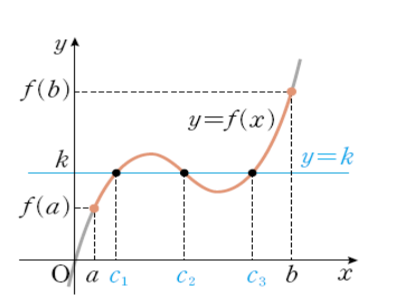
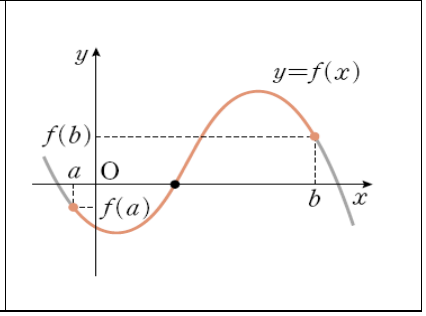
<tr><td></td><td></td><td>If f(b)>0 and f(a)<0, then there exists at least one solution.</td></tr>
<tr><td>
</td><td>
</td><td>
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Definition of the Average Rate of Change

The average rate of change of y (slope m) with respect to x over the interval $[a,b]$ is given by:

$$m=\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}=\frac{f(a+\Delta x)-f(a)}{\Delta x}=\frac{f(a+h)-f(a)}{h}$$

where $h=\Delta x=b-a$.

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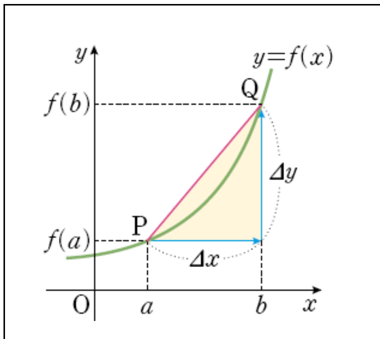
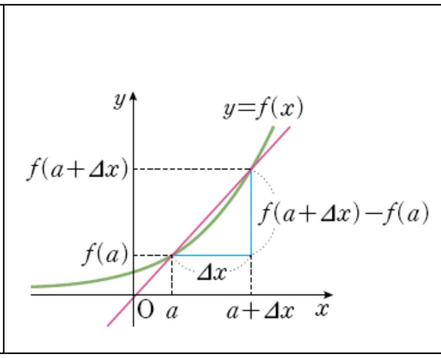
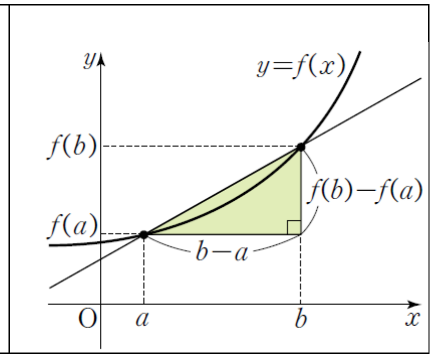
<tr><td>
</td><td>
</td><td>
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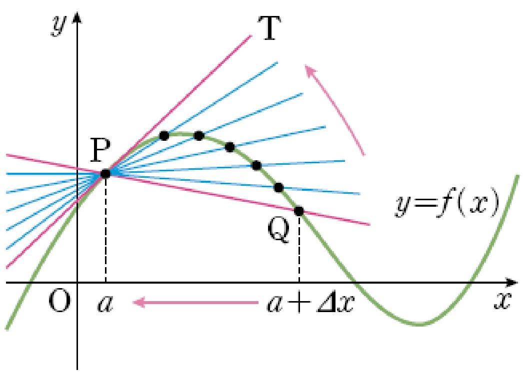
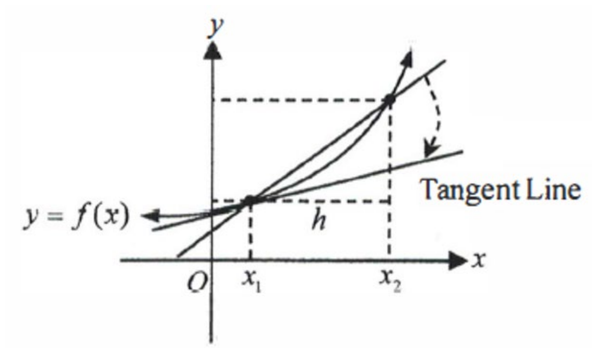
Definition of the Instant Rate of Change

The graph demonstrates the concept of secant lines approaching the tangent line at a specific point on a curve as the interval between the points on the secant line (Δx) approaches zero. This visual representation helps in understanding the definition of the derivative, which is the slope of the tangent line at a given point on the function.





$f'(a) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$	$f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

- **Secant Lines**: The lines passing through points P and Q are secant lines. These lines intersect the curve at two points and approximate the slope of the function between those points.

  - As Δx decreases (meaning Q moves closer to P), the secant lines approach the slope of the tangent line at P.

- **Tangent Line T**:

  - Definition: The tangent line at point P is the line that just touches the curve at P without crossing it. This line represents the instantaneous rate of change of the function at $x = a$.
  - Slope of the Tangent Line: The slope of the tangent line at P is the limit of the slopes of the secant lines as Δx approaches zero.

- The slope of the tangent line at $x = a$ is given by the derivative: (replace Δx to h)

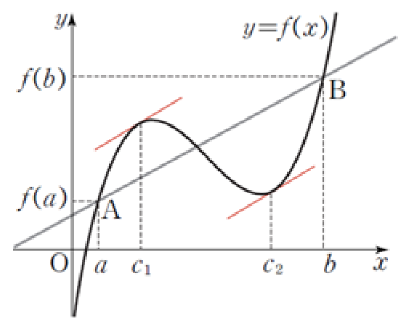
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

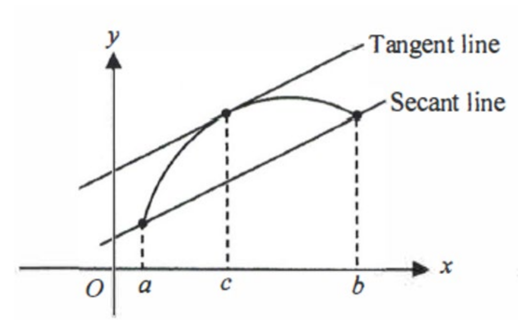
Theorem: The Mean-Value Theorem (MVT)

If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number c between a and b such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

The slope of the secant line is equal to the slope of the tangent line.





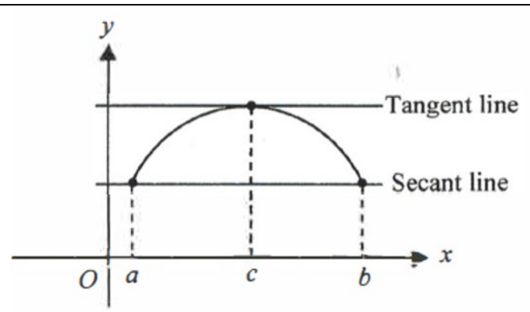
The Mean-Value Theorem guarantees that there is at least one point c in the interval (a, b) where the tangent line has the same slope as the secant line.

Theorem: Rolle's Theorem (Special case of the Mean Value Theorem)

Let f be continuous on the closed interval [a, b] and differentiable on the open interval (a, b).

If $f(b)=f(a)$, then there exists at least one number c between a and b such that

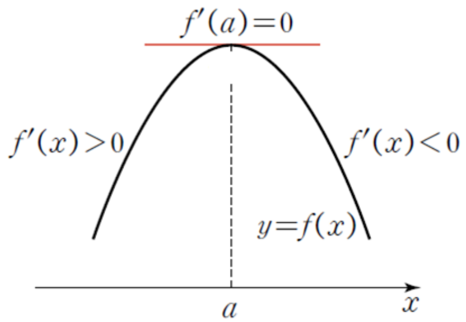
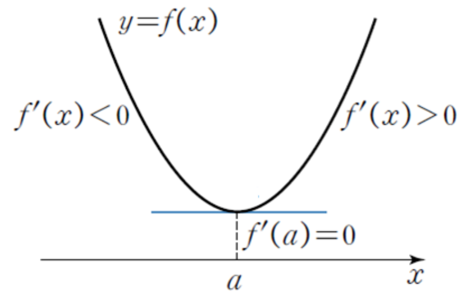
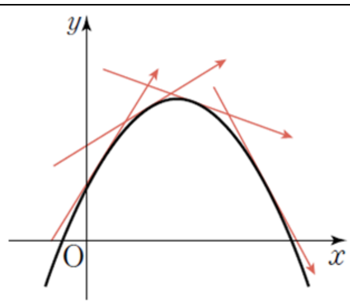
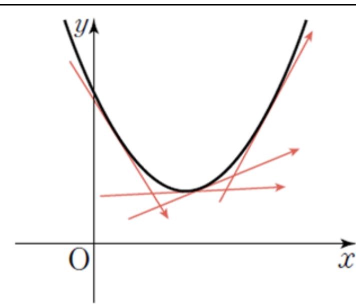
$$\frac{f(b)-f(a)}{b-a}=f'(c)=0.$$



Rolle's Theorem will guarantee the existence of an extreme value (relative maximum or relative minimum) in the interval.

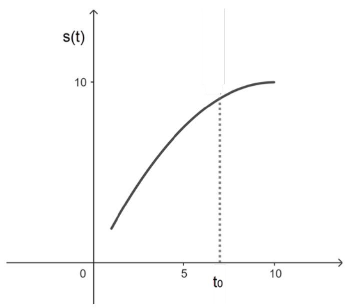
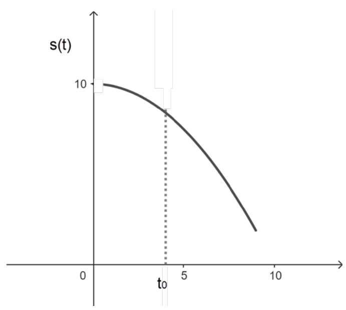
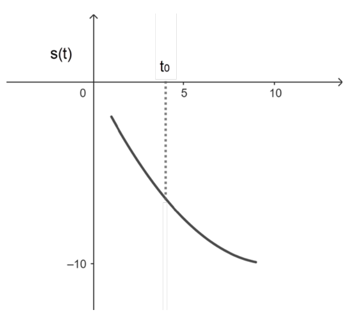
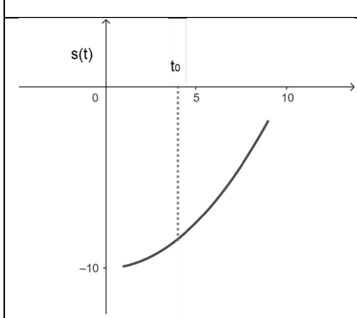
Maxima and Minima

- Maxima: At a local maximum, the derivative changes from positive to negative.
- Minima: At a local minimum, the derivative changes from negative to positive.
- Tangent lines at the critical points where $f'(a) = 0$ confirm the behavior of the slopes, showing a peak for maxima and a valley for minima.

Maxima (relative maximum at x=a)	Minima (relative minimum at x=a)
	
- Function $y = f(x)$: The curve represents the function. - Critical Point at a: The point where the slope of the tangent is zero, $f'(a) = 0$. - Left of a: $f'(x) > 0$, the function is increasing. - Right of a: $f'(x) < 0$, the function is decreasing. - This indicates a local maximum at $x = a$.	- Function $y = f(x)$: The curve represents the function. - Critical Point at a: The point where the slope of the tangent is zero, $f'(a) = 0$. - Left of a: $f'(x) < 0$, the function is decreasing. - Right of a: $f'(x) > 0$, the function is increasing. - This indicates a local minimum at $x = a$.
	
- Shows the tangent lines with positive slopes approaching $x = a$ from the left and negative slopes after $x = a$, confirming the maximum. - Concave Downward	- Shows the tangent lines with negative slopes approaching $x = a$ from the left and positive slopes after $x = a$, confirming the minimum. - Concave Upward

0-5. Behavior of the Particle about Position vs. Time Curve (Preview)

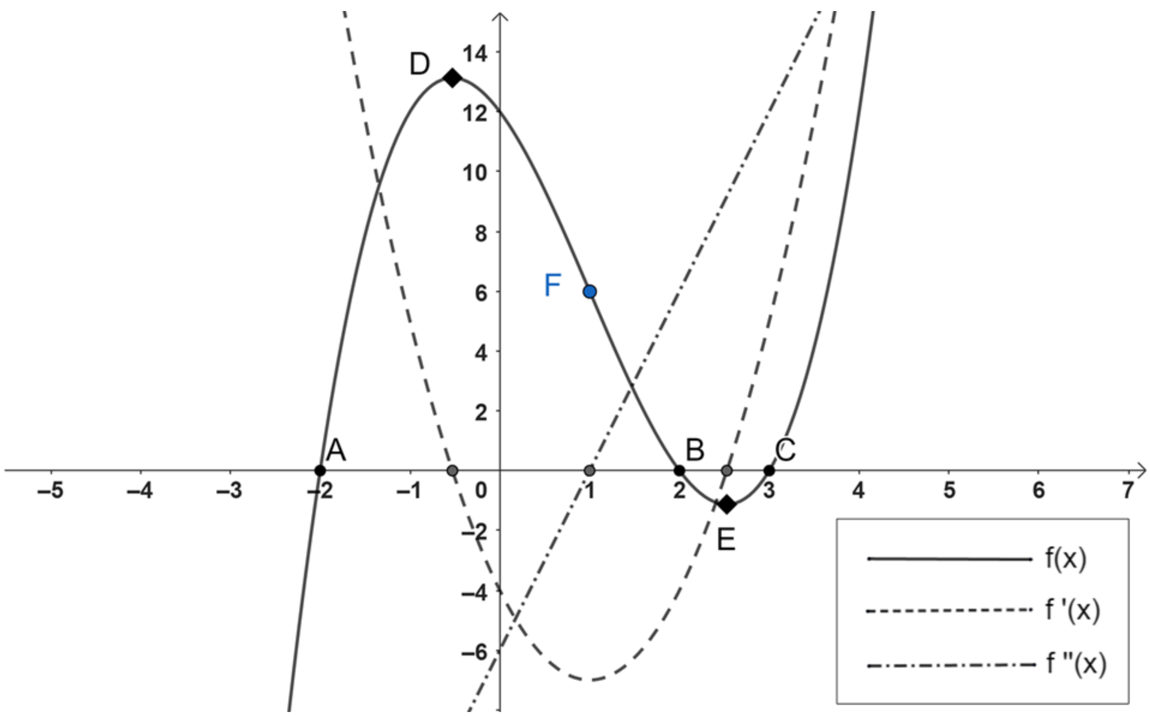
Observe behavior of the particle about the position versus time curve.

	At $t=t_0$ - Curve has positive slope. - Curve is concave down. - $s(t_0)>0$ - $s'(t_0)=v(t_0)>0$ - $s''(t_0)=a(t_0)<0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is decreasing. - Particle is slowing down. - $v(t_0)>0$ and $a(t_0)<0$
	2. At $t=t_0$ - Curve has negative slope. - Curve is concave down. - $s(t_0)>0$ - $s'(t_0)=v(t_0)<0$ - $s''(t_0)=a(t_0)<0$	- Particle is on the positive side of the origin. - Particle is moving in the negative direction. - Velocity is decreasing. - Particle is speeding up. - $v(t_0)<0$ and $a(t_0)<0$
	3. At $t=t_0$ - Curve has negative slope. - Curve is concave up. - $s(t_0)<0$ - $s'(t_0)=v(t_0)<0$ - $s''(t_0)=a(t_0)>0$	- Particle is on the negative side of the origin. - Particle is moving in the negative direction. - Velocity is increasing (**slope** is increasing by t is progressing). - Particle is slowing down. - $v(t_0)<0$ and $a(t_0)>0$
	4. At $t=t_0$ - Curve has positive slope. - Curve is concave up. - $s(t_0)<0$ - $s'(t_0)=v(t_0)>0$ - $s''(t_0)=a(t_0)>0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is increasing. - Particle is speeding up. - $v(t_0)>0$ and $a(t_0)>0$

0-6. Derivative Test (Preview)

Concept Expansion from Pre-Calculus:

1) Find local maximum (D) and local minimum (E) values and the inflection point (F) for $f(x) = x^3 - 3x^2 - 4x + 12$ without using Calculus Concept?



Solution

- We can easily find roots (A, B & C) by factorization $\Rightarrow roots: x = -2, 2, 3$
- However, it is not easy to find x values for D (local max), E(local min) and F(Inflection point). Before calculus, to solve this problem, we may need to use the approximation method.
- Once we learn about derivatives, then we can find these points easily.

2) Sketch the polynomial graph of $f(x) = x^3 - 3x^2 - 24x + 32$ by using $f'(x)$, $f''(x)$

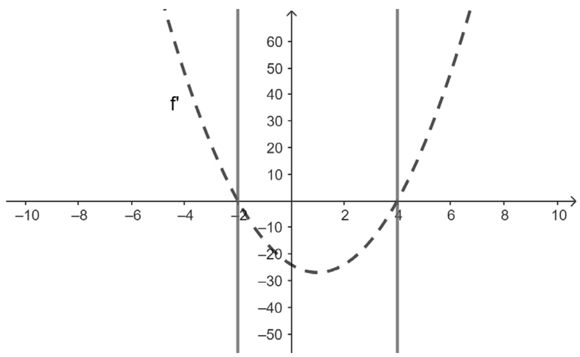
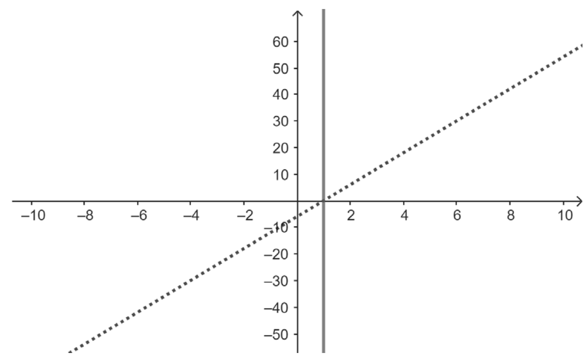
Solution Steps:

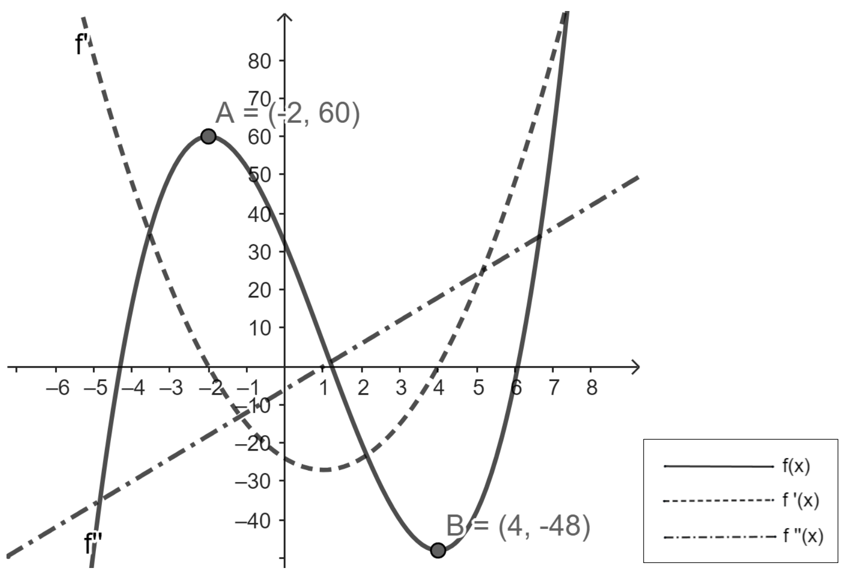
1. Find $f(x)$, $f'(x)$, $f''(x)$
   - $f(x) = x^3 - 3x^2 - 24x + 32$
   - $f'(x) = 3x^2 - 6x - 24$
   - $f''(x) = 6x - 6$
2. Find the first derivative ($f'(x)$) equal to zero to find **critical points** and its functional values if exist
   - $3x^2 - 6x - 24 = 0 \Rightarrow 3(x-4)(x+2) = 0 \Rightarrow x = 4, x = -2$
   - $f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 32 = 60$ (maxima)
   - $f(4) = (4)^3 - 3(4)^2 - 24(4) + 32 = -48$ (minima)
3. Set the second derivative **(** $f''(x)$ **)** equal to zero to find **inflection points** and its functional values if exist
   - $6x - 6 = 0 \Rightarrow 6(x-1) = 0 \Rightarrow x = 1$
   - $f(1) = (1)^3 - 3(1)^2 - 24(1) + 32 = 6$ (inflection point)
4. Determine the y-intercept
   - $f(0) = 0^3 - 3(0)^2 - 24(0) + 32 = 32$

5. Determine the **concavity and relative extrema** using the first and second derivatives
Number of critical points (including inflection points): 3, so need 4 sections on the graph

$f(x) = x^3 - 3x^2 - 24x + 32$							
$f'(x) = 3x^2 - 6x - 24$	**+**	Local Max $x = -2$	**-**	**-**	**-**	Local Min $x = 4$	**+**
	Increase	Critical Point	Decrease			Critical Point	Increase
$f''(x) = 6x - 6$	**-**	**-**	**-**	1	**+**	**+**	**+**
	Concave Down			Inflection Point	Concave Up		

6. Sketch the graph:

$f'(x)$	$f''(x)$
	



1. Plot the relative maximum at A $(-2,60)$.
2. Plot the relative minimum at B $(4,-48)$.
3. Plot the inflection point at C $(1,6)$.
4. Plot the y-intercept at D $(0,32)$.
5. Draw the curve concave down from $(-\infty,-2)$, then continue concave down through $(-2,60)$ to $(1,6)$.
6. Switch to concave up from $(1,6)$ to $(4,-48)$ and continue concave up to (∞,∞).

3) Sketch the rational function graph of $f(x)=\dfrac{x^2-4x+3}{x}$ by using $f'(x)$, $f''(x)$

To sketch the rational function $f(x)=\dfrac{x^2-4x+3}{x}$ using its first and second derivatives, follow these steps:

1. Simplify the Function: $f(x)=\dfrac{x^2-4x+3}{x}=x-4+\dfrac{3}{x}$

2. Find Asymptotes

- Vertical Asymptote: Occurs where the denominator is zero: $x=0$
- Slanted (oblique) Asymptote: $y=x-4$

3. Find Intercepts

- x-intercepts: Set $f(x)=0$: $x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$, So, $x=1$ and $x=3$.
- y-intercept: Set $x=0$: The function is undefined at $x=0$, so there is no y-intercept.

4. Find Critical Points (First Derivative)

- Find the first derivative $f'(x)$: $f'(x)=\dfrac{d}{dx}\left(x-4+\dfrac{3}{x}\right)=1-\dfrac{3}{x^2}$
- Set $f'(x)=0:1-\dfrac{3}{x^2}=0\ x=\pm\sqrt{3}$

5. Find Points of Inflection (Second Derivative)

- Find the second derivative $f''(x)$: $f''(x)=\dfrac{d}{dx}\left(1-\dfrac{3}{x^2}\right)=\dfrac{6}{x^3}$
- Set $f''(x)=0$: $\dfrac{6}{x^3}=0$
- There are no real solutions. So, there are no points of inflection.

6. Analyze Intervals of Increase/Decrease

- For $x>0$: $f'(x)=1-\frac{3}{x^2}$:
- If $x>\sqrt{3}$, $f'(x)>0$ (positive).
- If $0<x<\sqrt{3}$, $f'(x)<0$ (negative).
- For $x<0$: $f'(x)=1-\frac{3}{x^2}$:
- Always $f'(x)<0$ (negative).

So, $x=-\sqrt{3}$ and $x=\sqrt{3}$ are **critical points:**

- Increasing on $(\sqrt{3},\infty)$
- Decreasing on $(0,\sqrt{3})$ and $(-\infty,0)$

7. Sketch the Graph

- **Asymptotes:** Vertical asymptote at $x=0$, Horizontal asymptote at $y=1$
- **Intercepts:** x-intercepts at $x=1$ and $x=3$
- Critical Points:
  - Local Minimum at $x=\sqrt{3}$: $y=-0.5$
  - Local Maximum at $x=-\sqrt{3}$: $y=-7.5$

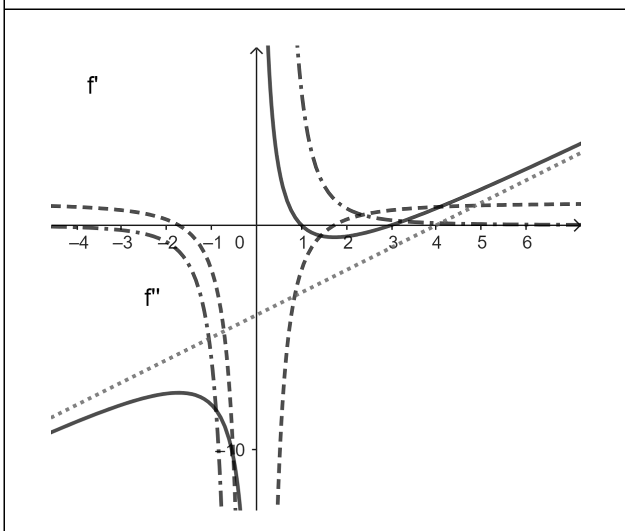
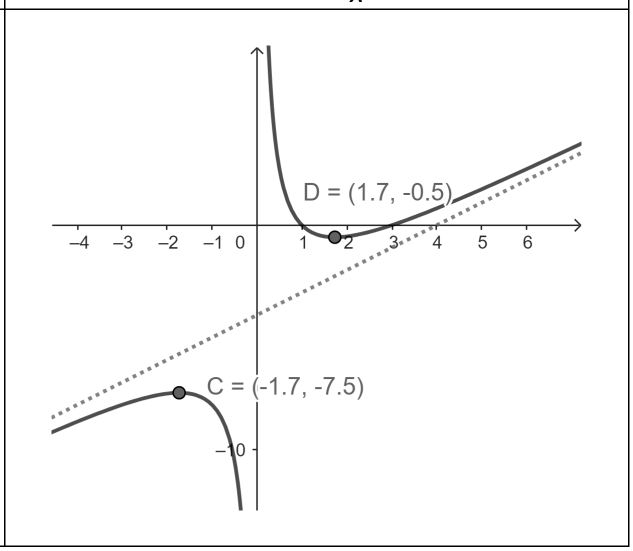
Alternate simple way by graphing $f'(x)$ & $f''(x)$

$$f(x)=\frac{x^2-3x-4}{x}$$

Find: (Refer below table)

- Find $f'(x)$, then find all critical points **candidates** by factorizing if exists
- Find $f''(x)$, then find all inflection points **candidates** by factorizing if exists
- Find signs (positivity or negativity) before and after (all candidates) critical/inflection points
- If $f'(x)\geq 0$, the $f(x)$ is increasing on the ranges
- If $f'(x)\leq 0$, the $f(x)$ is decreasing on the ranges
- If $f''(x)\geq 0$, the $f(x)$ is concave-up on the ranges
- If $f''(x)\leq 0$, the $f(x)$ is concave-down the ranges
- Find all critical/inflection points from candidates, then find $f(x)$ values (y-values)
- If local min/min y-values from critical points.

$f'(x)=1-\frac{3}{x^2}$	+	Local Max $x=\sqrt{3}$	-	0 (undefined)	-	Local Min $x=-\sqrt{3}$	+
	Increase	Critical Point	decrease	Not critical point	decrease	Critical Point	Increase
$f''(x)=\frac{6}{x^3}$	-	-	-	0 (undefined)	+	+	+
	Concave Down			Inflection Point	Concave Down		

$f'(x)=1-\frac{3}{x^2}$, $f''(x)=\frac{6}{x^3}$	$f(x)=\frac{x^2-3x-4}{x}$
	

0-7. Derivative Formula (Preview)

1) Derivative and Integral Rules

	Derivative	Integral (Antiderivative)
1	$\frac{d}{dx}n=0$	$\int 0dx=C$
2	$\frac{d}{dx}x=1$	$\int 1dx=x+C$
3	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\int\left[x^n\right]dx=\frac{x^{n+1}}{n+1}+C$
4	$\frac{d}{dx}\left[e^x\right]=e^x$	$\int\left[e^x\right]dx=e^x+C$
5	$\frac{d}{dx}[\ln x]=\frac{1}{x}$	$\int\left[\frac{1}{x}\right]dx=\ln x+C$
6	$\frac{d}{dx}\left[n^x\right]=n^x\ln n$	$\int\left[n^x\right]dx=\frac{n^x}{\ln n}+C$
7	$\frac{d}{dx}[\sin x]=\cos x$	$\int[\cos x]dx=\sin x+C$
8	$\frac{d}{dx}[\cos x]=-\sin x$	$\int[\sin x]dx=-\cos x+C$
9	$\frac{d}{dx}[\tan x]=\sec^2 x$	$\int\left[\sec^2 x\right]dx=\tan x+C$
10	$\frac{d}{dx}[\cot x]=-\csc^2 x$	$\int\left[\csc^2 x\right]dx=-\cot x+C$
11	$\frac{d}{dx}[\sec x]=\sec x\tan x$	$\int[\tan x\sec x]dx=\sec x+C$
12	$\frac{d}{dx}[\csc x]=-\csc x\cot x$	$\int[\cot x\csc x]dx=-\csc x+C$
13	$\frac{d}{dx}[\arcsin x]=\frac{1}{\sqrt{1-x^2}}$	$\int\frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C$
14	$\frac{d}{dx}[\arccos x]=-\frac{1}{\sqrt{1-x^2}}$	$\int\frac{1}{\sqrt{1-x^2}}dx=\arccos x+C$
15	$\frac{d}{dx}[\arctan x]=\frac{1}{1+x^2}$	$\int\frac{1}{1+x^2}dx=\arctan x+C$

16	$\frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \operatorname{arccot} x + C$
18	$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$
19	$\frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arccsc} x + C$

2) General Differentiation Rules

Let **c** be a real number, **n** be a rational number, **u** and **v** be differentiable functions of **x**, let **f** be a differentiable function of **u**, and let **a** be a positive real number ($a \neq 1$).

	Differentiation Rules	
1	Constant Rule	$\frac{d}{dx}[c] = 0$
2	Constant Multiple Rule	$\frac{d}{dx}[cu] = cu'$
3	Product Rule	$\frac{d}{dx}[uv] = uv' + vu'$
4	Chain Rule	$\frac{d}{dx}[f(u)] = f'(u)u'$
5	(Simple) Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$
6	Sum or Difference Rule	$\frac{d}{dx}[u \pm v] = u' \pm v'$
7	Quotient Rule	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
8	General Power Rule	$\frac{d}{dx}[u^n] = nu^{n-1}u'$
9	Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\tan x] = \sec^2 x$ $\frac{d}{dx}[\cot x] = -\csc^2 x$

		$\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$
10	Derivatives of Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$ $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$ $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
11	Derivatives of Inverse Trigonometric Functions	$\frac{d}{dx}\left[\arcsin x\right] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\arccos x\right] = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\left[\arctan x\right] = \frac{1}{1+x^2}$ $\frac{d}{dx}\left[\text{arccot}\, x\right] = -\frac{1}{1+x^2}$ $\frac{d}{dx}\left[\text{arcsec}\, x\right] = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}\left[\text{arccsc}\, x\right] = -\frac{1}{x\sqrt{x^2-1}}$
12	Derivatives of Inverse Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$ $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$ $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{\lvert u \rvert\sqrt{u^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, u] = \frac{-u'}{\lvert u \rvert\sqrt{u^2-1}}$

13	Derivatives of Basic Hyperbolic Trigonometric Functions $\sinh(x)=\frac{e^x-e^{-x}}{2}$ $\cosh(x)=\frac{e^x+e^{-x}}{2}$	$\frac{d}{dx}[\sinh(x)]=\cosh(x)$ $\frac{d}{dx}[\cosh(x)]=\sinh(x)$ $\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$ $\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$ $\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$ $\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$
14	Derivatives of Inverse Hyperbolic Trigonometric Functions	$\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$ $\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$ $\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$ $\frac{d}{dx}[\text{sech}^{-1}(x)]=\frac{-1}{x\sqrt{1-x^2}}$ $\frac{d}{dx}[\text{csch}^{-1}(x)]=\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$ $\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$
15	Derivatives of Exponential and Logarithmic Functions	$\frac{d}{dx}[e^x]=e^x$ $\frac{d}{dx}[\ln x]=\frac{1}{x}$ $\frac{d}{dx}[a^x]=(\ln a)a^x$ $\frac{d}{dx}[\log_a x]=\frac{1}{(\ln a)x}$
16	Basic Differentiation Rules for Elementary Functions (**u & v** be differentiable functions of **x)**	$\frac{d}{dx}[u^n]=nu^{n-1}u'$ $\frac{d}{dx}[\lvert u\rvert]=\frac{u}{\lvert u\rvert}u',\quad u\neq 0$ $\frac{d}{dx}[\ln u]=\frac{u'}{u}$ $\frac{d}{dx}[e^u]=e^u u'$

		$\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$ $\frac{d}{dx}[a^u] = (\ln a)a^u u'$

3) Hyperbolic functions are analogs of the circular trigonometric functions, but for a hyperbola. They are extensively used in various areas of mathematics, including algebra, calculus, and complex analysis. Here are the basic hyperbolic functions along with their definitions:

1	Hyperbolic Sine ($\sinh x$)	$\sinh x = \frac{e^x - e^{-x}}{2}$
2	Hyperbolic Cosine ($\cosh x$)	$\cosh x = \frac{e^x + e^{-x}}{2}$
3	Hyperbolic Tangent ($\tanh x$)	$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4	Hyperbolic Cosecant ($\text{csch } x$)	$\text{csch } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
5	Hyperbolic Secant ($\text{sech } x$)	$\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
6	Hyperbolic Cotangent ($\coth x$)	$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

4) List of antiderivative Formulas: covering a wider range of functions. These include basic functions, exponential and logarithmic functions, trigonometric functions, and some of their inverses

	Functions	**Antiderivative Formulas:**
1	Constant Function	$\int a\,dx = ax + C$
2	Power Function	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3	Exponential Function	$\int e^x\,dx = e^x + C$
4	General Exponential Function	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1)$
5	Natural Logarithm	$\int \frac{1}{x} dx = \ln\lvert x \rvert + C$

6	Sine Functions	$\int \sin(x)dx = -\cos(x) + C$
7	Cosine Functions	$\int \cos(x)dx = \sin(x) + C$
8	Tangent Functions	$\int \tan(x)dx = -\ln\|\cos(x)\| + C$
9	Cotangent (cot) Functions	$\int \cot(x)dx = \ln\|\sin(x)\| + C$
10	Secant (sec) Functions	$\int \sec(x)dx = \ln\|\sec(x) + \tan(x)\| + C$
11	Cosecant (csc) Functions	$\int \csc(x)dx = -\ln\|\csc(x) + \cot(x)\| + C$
12	Inverse Sine (arcsin) Functions	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$
13	Inverse Tangent (arctan) Functions	$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$
14	sinh (Hyperbolic Sine) Functions	$\int \sinh(x)dx = \cosh(x) + C$
15	cosh (Hyperbolic Cosine) Functions	$\int \cosh(x)dx = \sinh(x) + C$
16	Integral of sec^2	$\int \sec^2(x)dx = \tan(x) + C$
17	Integral of csc^2	$\int \csc^2(x)dx = -\cot(x) + C$
18	Integral of $\sec(x)\tan(x)$	$\int \sec(x)\tan(x)dx = \sec(x) + C$
19	Integral of $\csc(x)\cot(x)$	$\int \csc(x)\cot(x)dx = -\csc(x) + C$

0-8. Find Derivatives (Preview)

Example: Solve all

1) Find the derivative of the function $f(x)=7$. (Constant Rule)

- Using the constant rule, which states $\frac{d}{dx}[c]=0$
- $f'(x)=\frac{d}{dx}[7]=0$

2) Find the derivative of the function $f(x)=5x^3$. (Constant Multiple Rule)

- Using the constant multiple rule, which states $\frac{d}{dx}[cu]=cu'$
- $f'(x)=\frac{d}{dx}[5x^3]=5\cdot\frac{d}{dx}[x^3]=5\cdot 3x^2=15x^2$

3) Find the derivative of the function $f(x)=x^2\sin(x)$. (Product Rule)

- Using the product rule, which states $\frac{d}{dx}[uv]=uv'+vu'$
- $u=x^2$, $v=\sin(x)$
- $u'=2x$, $v'=\cos(x)$
- $f'(x)=(x^2)'\sin(x)+x^2(\sin(x))'=2x\sin(x)+x^2\cos(x)$

4) Find the derivative of the function $f(x)=\sin(3x)$. (Chain Rule)

- Using the chain rule, which states $\frac{d}{dx}[f(u)] = f'(u)u'$
- $f(u) = \sin(u)$, $u = 3x$
- $f'(u) = \cos(u)$, $u' = 3$
- $f'(x) = \cos(3x) \cdot 3 = 3\cos(3x)$

5) Find the derivative of the function $f(x) = x^5$. ((Simple) Power Rule)

- Using the power rule, which states $\frac{d}{dx}[x^n] = nx^{n-1}$
- $f'(x) = \frac{d}{dx}[x^5] = 5x^4$

6) Find the derivative of the function $f(x) = x^3 - 4x + 7$. (Sum or Difference Rule)

- Using the sum or difference rule, which states $\frac{d}{dx}[u \pm v] = u' \pm v'$
- $f'(x) = \frac{d}{dx}[x^3] - \frac{d}{dx}[-4x] + \frac{d}{dx}[7] = 3x^2 - 4 + 0 = 3x^2 - 4$

7) Find the derivative of the function $f(x) = \frac{x^2}{\sin(x)}$. (Quotient Rule)

- Using the quotient rule, which states $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
- $u = x^2$, $v = \sin(x)$
- $u' = 2x$, $v' = \cos(x)$
- $f'(x) = \frac{\sin(x) \cdot 2x - x^2 \cdot \cos(x)}{\sin^2(x)} = \frac{2x\sin(x) - x^2\cos(x)}{\sin^2(x)}$

8) Find the derivative of the function $f(x)=(3x^2+2)^4$. (General Power Rule)

- Using the general power rule, which states $\frac{d}{dx}[u^n]=nu^{n-1}u'$
- $u=3x^2+2$, $u'=6x$
- $n=4$
- $f'(x)=4(3x^2+2)^3\cdot 6x=24x(3x^2+2)^3$

9) Find the derivative of the function $f(x)=\tan(x)$. (Derivatives of Trigonometric Functions)

- Using the derivative rule for the tangent function, which states $\frac{d}{dx}[\tan(x)]=\sec^2(x)$
- $f'(x)=\frac{d}{dx}[\tan(x)]=\sec^2(x)$

10) Find the derivative of the function $f(x)=\sin(x)$. (Derivative of $\sin(x)$)

- Using the derivative rule for the sine function, which states $\frac{d}{dx}[\sin(x)]=\cos(x)$
- $f'(x)=\frac{d}{dx}[\sin(x)]=\cos(x)$

11) Find the derivative of the function $f(x)=\cos(x)$. (Derivative of $\cos(x)$)

- Using the derivative rule for the cosine function, which states $\frac{d}{dx}[\cos(x)]=-\sin(x)$
- $f'(x)=\frac{d}{dx}[\cos(x)]=-\sin(x)$

12) Find the derivative of the function $f(x) = \tan(2x)$. (Derivative of $\tan(x)$)

- Using the derivative rule for the tangent function, which states $\frac{d}{dx}[\tan(u)] = \sec^2(u) \cdot u'$
- $f'(x) = \frac{d}{dx}[\tan(2x)] = \sec^2(2x) \cdot 2$

13) Find the derivative of the function $f(x) = \cot(x)$. (Derivative of $\cot(x)$)

- Using the derivative rule for the cotangent function, which states $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$
- $f'(x) = \frac{d}{dx}[\cot(x)] = -\csc^2(x)$

14) Find the derivative of the function $f(x) = \sec(x)$. (Derivative of $\sec(x)$)

- Using the derivative rule for the secant function, which states $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
- $f'(x) = \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$

15) Find the derivative of the function $f(x) = \csc(x)$. (Derivative of $\csc(x)$)

- Using the derivative rule for the cosecant function, which states $\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$
- $f'(x) = \frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$

16) Find the derivative of the function $f(x) = \sin(3x^2 + 2x)$. (Derivative of $\sin(u)$ where u is a function of x)

- Using the chain rule and the derivative of sine, which states $\frac{d}{dx}[\sin(u)] = (\cos(u))u'$
- $u = 3x^2 + 2x$, $u' = 6x + 2$
- $f'(x) = \cos(3x^2 + 2x) \cdot (6x + 2) = (6x + 2)\cos(3x^2 + 2x)$

17) Find the derivative of the function $f(x) = \cos(x^3 - x)$. (Derivative of $\cos(u)$ where u is a function of x)

- Using the chain rule and the derivative of cosine, which states $\frac{d}{dx}[\cos(u)] = -(\sin(u))u'$
- $u = x^3 - x$, $u' = 3x^2 - 1$
- $f'(x) = -\sin(x^3 - x) \cdot (3x^2 - 1) = -(3x^2 - 1)\sin(x^3 - x)$

18) Find the derivative of the function $f(x) = \tan(2x^2 - 3x)$. (Derivative of $\tan(u)$ where u is a function of x)

- Using the chain rule and the derivative of tangent, which states $\frac{d}{dx}[\tan(u)] = (\sec^2(u))u'$
- $u = 2x^2 - 3x$, $u' = 4x - 3$
- $f'(x) = \sec^2(2x^2 - 3x) \cdot (4x - 3) = (4x - 3)\sec^2(2x^2 - 3x)$

19) Find the derivative of the function $f(x) = \cot(4x^3 + x^2)$. (Derivative of $\cot(u)$ where u is a function of x)

- Using the chain rule and the derivative of cotangent, which states $\frac{d}{dx}[\cot(u)] = -(\csc^2(u))u'$

- $u = 4x^3 + x^2$, $u' = 12x^2 + 2x$
- $f'(x) = -\csc^2(4x^3 + x^2)\cdot(12x^2 + 2x) = -(12x^2 + 2x)\csc^2(4x^3 + x^2)$

20) Find the derivative of the function $f(x) = \sec(3x^2 + x)$. (Derivative of $\sec(u)$ where u is a function of x)

- Using the chain rule and the derivative of secant, which states $\frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))u'$
- $u = 3x^2 + x$, $u' = 6x + 1$
- $f'(x) = \sec(3x^2 + x)\tan(3x^2 + x)\cdot(6x + 1) = (6x + 1)\sec(3x^2 + x)\tan(3x^2 + x)$

21) Find the derivative of the function $f(x) = \csc(x^2 + 2x)$. (Derivative of $\csc(u)$ where u is a function of x)

- Using the chain rule and the derivative of cosecant, which states $\frac{d}{dx}[\csc(u)] = -(\csc(u)\cot(u))u'$
- $u = x^2 + 2x$, $u' = 2x + 2$
- $f'(x) = -\csc(x^2 + 2x)\cot(x^2 + 2x)\cdot(2x + 2) = -(2x + 2)\csc(x^2 + 2x)\cot(x^2 + 2x)$

22) Find the derivative of the function $f(x) = \sinh(x)$. (Derivative of $\sinh(x)$)

- Using the derivative rule for the hyperbolic sine function, which states $\frac{d}{dx}[\sinh(x)] = \cosh(x)$
- $f'(x) = \frac{d}{dx}[\sinh(x)] = \cosh(x)$

23) Find the derivative of the function $f(x)=\cosh(x)$. (Derivative of $\cosh(x)$)

- Using the derivative rule for the hyperbolic cosine function, which states $\frac{d}{dx}[\cosh(x)]=\sinh(x)$
- $f'(x)=\frac{d}{dx}[\cosh(x)]=\sinh(x)$

24) Find the derivative of the function $f(x)=\tanh(x)$. (Derivative of $\tanh(x)$)

- Using the derivative rule for the hyperbolic tangent function, which states $\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$
- $f'(x)=\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$

25) Find the derivative of the function $f(x)=\text{sech}(x)$. (Derivative of $\text{sech}(x)$)

- Using the derivative rule for the hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$
- $f'(x)=\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$

26) Find the derivative of the function $f(x)=\text{csch}(x)$. (Derivative of $\text{csch}(x)$)

- Using the derivative rule for the hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$
- $f'(x)=\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$

27) Find the derivative of the function $f(x)=\coth(x)$. (Derivative of $\coth(x)$)

- Using the derivative rule for the hyperbolic cotangent function, which states $\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$
- $f'(x)=\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$

28) Find the derivative of the function $f(x)=\sinh^{-1}(x)$. (Derivative of $\sinh^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic sine function, which states $\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$
- $f'(x)=\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$

29) Find the derivative of the function $f(x)=\cosh^{-1}(x)$. (Derivative of $\cosh^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic cosine function, which states $\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$
- $f'(x)=\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$

30) Find the derivative of the function $f(x)=\tanh^{-1}(x)$. (Derivative of $\tanh^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic tangent function, which states $\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$

- $f'(x) = \frac{d}{dx}[\tanh^{-1}(x)] = \frac{1}{1-x^2}$

31) Find the derivative of the function $f(x) = \text{sech}^{-1}(x)$. (Derivative of $\text{sech}^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$
- $f'(x) = \frac{d}{dx}[\text{sech}^{-1}(x)] = \frac{-1}{x\sqrt{1-x^2}}$

32) Find the derivative of the function $f(x) = \text{csch}^{-1}(x)$. (Derivative of $\text{csch}^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{|x|\sqrt{1+x^2}}$
- $f'(x) = \frac{d}{dx}[\text{csch}^{-1}(x)] = \frac{-1}{|x|\sqrt{1+x^2}}$

33) Find the derivative of the function $f(x) = \coth^{-1}(x)$. (Derivative of $\coth^{-1}(x)$)

- Using the derivative rule for the inverse hyperbolic cotangent function, which states $\frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$
- $f'(x) = \frac{d}{dx}[\coth^{-1}(x)] = \frac{1}{1-x^2}$

34) Find the derivative of the function $f(x) = e^x$. (Derivative of e^x)

- Using the derivative rule for the exponential function, which states $\frac{d}{dx}[e^x] = e^x$
- $f'(x) = \frac{d}{dx}[e^x] = e^x$

35) Find the derivative of the function $f(x) = \ln(x)$. (Derivative of $\ln(x)$)

- Using the derivative rule for the natural logarithm function, which states $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$
- $f'(x) = \frac{d}{dx}[\ln(x)] = \frac{1}{x}$

36) Find the derivative of the function $f(x) = 2^x$. (Derivative of a^x)

- Using the derivative rule for the exponential function with base a, which states $\frac{d}{dx}[a^x] = (\ln a)a^x$
- $f'(x) = \frac{d}{dx}[2^x] = (\ln 2)2^x$

37) Find the derivative of the function $f(x) = \log_2(x)$. (Derivative of $\log_a(x)$)

- Using the derivative rule for the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(x)] = \frac{1}{(\ln a)x}$
- $f'(x) = \frac{d}{dx}[\log_2(x)] = \frac{1}{(\ln 2)x}$

38) Find the derivative of the function $f(x) = (3x^2 + 2)^5$. (Derivative of u^n)

- Using the general power rule, which states $\frac{d}{dx}[u^n] = nu^{n-1}u'$
- $u = 3x^2 + 2$, $u' = 6x$
- $f'(x) = 5(3x^2 + 2)^4 \cdot 6x = 30x(3x^2 + 2)^4$

39) Find the derivative of the function $f(x) = |3x - 4|$. (Derivative of |u|)

- Using the rule for the derivative of the absolute value function, which states $\frac{d}{dx}[|u|] = \frac{u}{|u|}u'$ where $u \neq 0$
- $u = 3x - 4$, $u' = 3$
- $f'(x) = \frac{3x-4}{|3x-4|} \cdot 3 = \frac{3(3x-4)}{|3x-4|}$

40) Find the derivative of the function $f(x) = \ln(2x^3 + 5)$. (Derivative of $\ln(u)$)

- Using the rule for the derivative of the natural logarithm function, which states $\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$
- $u = 2x^3 + 5$, $u' = 6x^2$
- $f'(x) = \frac{6x^2}{2x^3 + 5}$

41) Find the derivative of the function $f(x) = e^{4x^2}$. (Derivative of e^u)

- Using the rule for the derivative of the exponential function, which states $\frac{d}{dx}[e^u] = e^u u'$
- $u = 4x^2$, $u' = 8x$
- $f'(x) = e^{4x^2} \cdot 8x = 8xe^{4x^2}$

0-9. Find Antiderivatives (Preview)

Example: Solve all

1) Find the antiderivative of $\int 6dx$.

- Using the antiderivative formula for a constant function, $\int adx = ax + C$

2) Find the antiderivative of $\int x^3 dx$.

- Using the power rule for antiderivatives, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where $n \neq -1$
- $\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$

3) Find the antiderivative of $\int e^x dx$.

- Using the antiderivative formula for the exponential function, $\int e^x dx = e^x + C$

4) Find the antiderivative of $\int 3^x dx$.

- Using the antiderivative formula for the general exponential function, $\int a^x dx = \frac{a^x}{\ln(a)} + C$ where $a > 0, a \neq 1$
- $\int 3^x dx = \frac{3^x}{\ln(3)} + C$

5) Find the antiderivative of $\int \frac{1}{x} dx$.

- Using the antiderivative formula for the natural logarithm, $\int \frac{1}{x} dx = \ln|x| + C$

6) Find the antiderivative of $\int \sin(x) dx$.

- Using the antiderivative formula for the sine function, $\int \sin(x) dx = -\cos(x) + C$

7) Find the antiderivative of $\int \cos(x) dx$.

- Using the antiderivative formula for the cosine function, $\int \cos(x) dx = \sin(x) + C$

8) Find the antiderivative of $\int \tan(x) dx$.

- Using the antiderivative formula for the tangent function, $\int \tan(x) dx = -\ln|\cos(x)| + C$,

9) Find the antiderivative of $\int \cot(x) dx$.

- Using the antiderivative formula for the cotangent function, $\int \cot(x) dx = \ln|\sin(x)| + C$

10) Find the antiderivative of $\int \sec(x)dx$ **.**

- Using the antiderivative formula for the secant function, $\int \sec(x)dx = \ln|\sec(x)+\tan(x)| + C$

11) Find the antiderivative of $\int \csc(x)dx$ **.**

- Using the antiderivative formula for the cosecant function, $\int \csc(x)dx = -\ln|\csc(x)+\cot(x)| + C$

12) Find the antiderivative of $\int \frac{1}{\sqrt{1-x^2}}dx$ **.**

- Using the antiderivative formula for the inverse sine function, $\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$

13) Find the antiderivative of $\int \frac{1}{1+x^2}dx$ **.**

- Using the antiderivative formula for the inverse tangent function, $\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$

14) Find the antiderivative of $\int \sinh(x)dx$ **.**

- Using the antiderivative formula for the hyperbolic sine function, $\int \sinh(x)dx = \cosh(x) + C$

15) Find the antiderivative of $\int \cosh(x)dx$.

- Using the antiderivative formula for the hyperbolic cosine function, $\int \cosh(x)dx = \sinh(x) + C$

16) Find the antiderivative of $\int \sec^2(x)dx$.

- Using the antiderivative formula for the integral of $\sec^2(x)$, $\int \sec^2(x)dx = \tan(x) + C$

17) Find the antiderivative of $\int \csc^2(x)dx$.

- Using the antiderivative formula for the integral of $\csc^2(x)$, $\int \csc^2(x)dx = -\cot(x) + C$

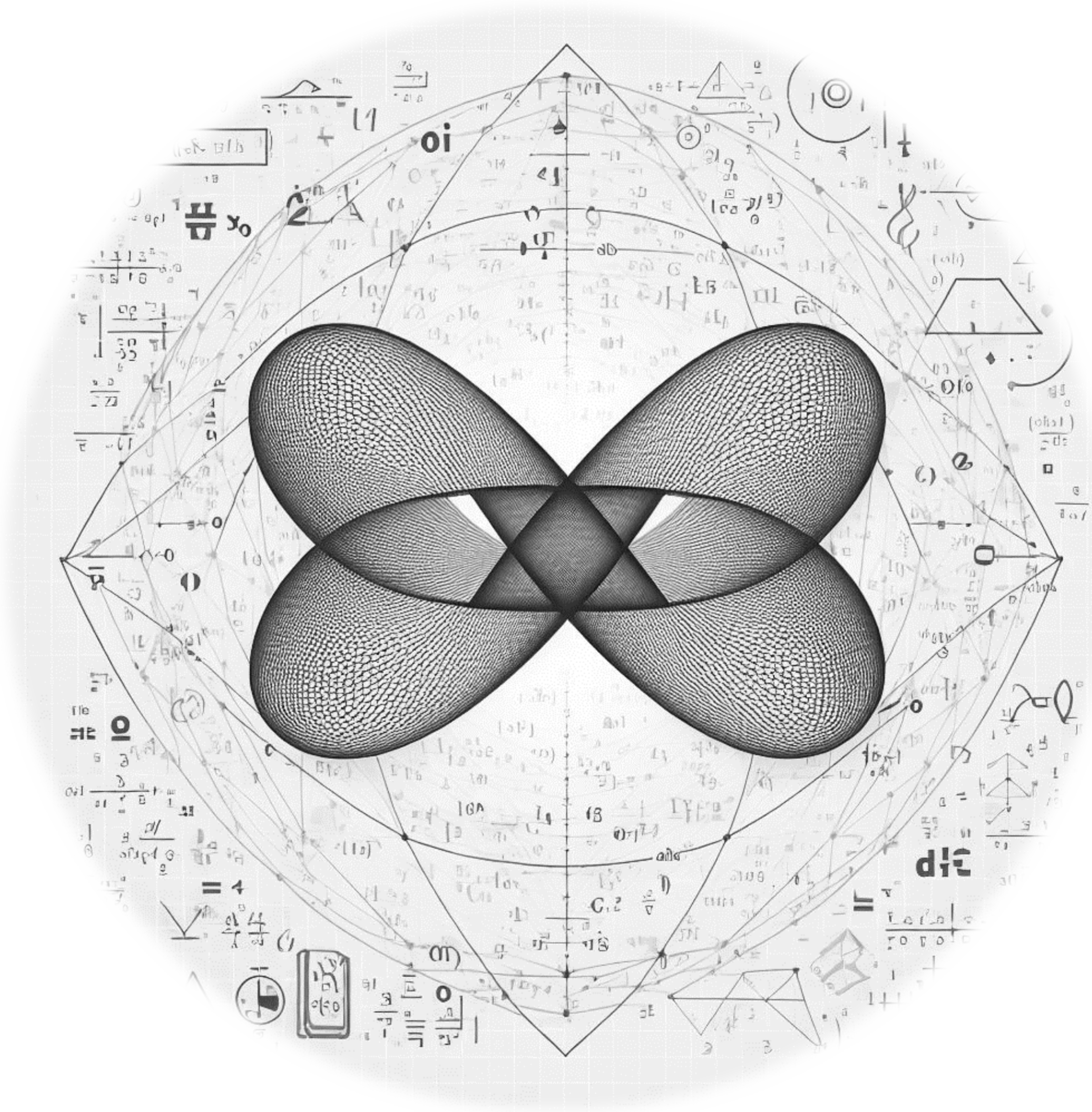
18) Find the antiderivative of $\int \sec(x)\tan(x)dx$.

- Using the antiderivative formula for the integral of $\sec(x)\tan(x)$, $\int \sec(x)\tan(x)dx = \sec(x) + C$

19) Find the antiderivative of $\int \csc(x)\cot(x)dx$.

- Using the antiderivative formula for the integral of $\csc(x)\cot(x)$, $\int \csc(x)\cot(x)dx = -\csc(x) + C$

Chapter 1. Limits and Continuity for Point Limits



1-1. Introduction to Limits (Point Limits)

The **concept of limits** is foundational in calculus and involves approaching a particular point on the function.

A limit describes the behavior of a function as it approaches a certain value (x-value), regardless of what the function's value is exactly at that point.

This concept is crucial for dealing with situations where the function becomes difficult or impossible to evaluate directly at that point due to discontinuities or undefined expressions.

A **point limit** is specifically the value that a function approaches as the input (or x-value) approaches a particular point. It's expressed as:

$$\lim_{x \to a} f(x) = L$$

This notation means that as x gets closer and closer to a, $f(x)$ gets arbitrarily close to L.

Formulas:

- Definition of a Limit: $\lim_{x \to a} f(x) = L$ means for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

- One-sided Limits:
    - Right-hand limit: $\lim_{x \to a^+} f(x) = L$
    - Left-hand limit: $\lim_{x \to a^-} f(x) = L$

Examples:

1) Calculate $\lim_{x\to 2}(3x+1)$ **.**

- Solve Directly substitute $x=2$ into the function.
- $\lim_{x\to 2}(3x+1)=3(2)+1=7$

2) Find $\lim_{x\to 0}\frac{\sin x}{x}$ **.**

- Solve This limit is a standard limit in calculus and equals 1. It demonstrates that as x approaches 0, the ratio of $\sin x$ to x approaches 1, despite the function being undefined at $x=0$.

3) Evaluate the limit: $\lim_{x\to 3}(2x+1)$ **.**

- To find the limit as x approaches 3, substitute 3 into the function: $2(3)+1=6+1=7$
- Therefore, $\lim_{x\to 3}(2x+1)=7$.

4) Find the limit: $\lim_{x\to 0}\frac{x^2-1}{x-1}$ **.**

- Factoring the numerator, we get: $\lim_{x\to 0}\frac{(x+1)(x-1)}{x-1}$
- The $x-1$ terms cancel out, giving: $\lim_{x\to 0}(x+1)=0+1=1$

1-2. Evaluating Limits Using Direct Substitution

Direct substitution is a straightforward method for evaluating limits in calculus.

It involves substituting the point at which the limit is taken directly into the function, assuming the function is continuous at that point.

This method is particularly useful and efficient when the function **does not exhibit (DNE)** any indeterminate forms like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Steps for Using Direct Substitution

1. Substitute the Limit Point into the Function:

- Plug the value $x = a$ into the function $f(x)$ directly.

2. Evaluate the Function:

- If the function evaluates to a finite number, that is the limit.
- If it results in an undefined expression or indeterminate form, alternative methods like factoring, rationalizing, or **L'Hopital's rule may be required**.

When to Use Direct Substitution

- Function is Continuous at the Point: If f is known to be continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$.
- No Indeterminate Forms are Encountered: If plugging in $x = a$ gives a definite value (not forms like $0/0$ or ∞/∞).

Significance

- Direct substitution is often the first method tried in limit calculations due to its simplicity and quick application.
- It is a fundamental tool for assessing the behavior of functions as they approach particular points.

Applications

- Mathematics Education: Teaching basic calculus concepts and introducing limit calculations.
- Engineering and Physics: Quick checks for continuous behavior in equations modeling physical phenomena.
- Economics: Evaluating economic models at specific points for analysis and forecasting.

Tips for Effective Application

- Always verify the continuity of the function at the point of interest before using direct substitution.
- Be aware of the function's domain to avoid undefined behavior.
- If direct substitution results in an indeterminate form, consider other techniques for evaluating the limit.

Examples:

1) Calculate $\lim_{x\to 3}(2x+1)$:

- Step 1: Substitute the Limit Point into the Function $f(x)=2x+1$
  Substitute $x=3$: $f(3)=2(3)+1=7$
- Step 2: Evaluate the Function
  The function evaluates directly to 7, hence: $\lim_{x\to 3}(2x+1)=7$

2) Evaluate the limit: $\lim_{x \to 0} \frac{\sqrt{x+4}-2}{x}$.

- Direct substitution gives which is indeterminate $0/0$.
- To resolve this, rationalize the numerator: $\lim_{x \to 0} \frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}$
- This simplifies to: $\lim_{x \to 0} \frac{x}{x(\sqrt{x+4}+2)} = \lim_{x \to 0} \frac{1}{\sqrt{x+4}+2}$
- Now direct substitution yields: $\frac{1}{\sqrt{0+4}+2} = \frac{1}{4}$

3) Determine $\lim_{x \to \infty} \frac{\sqrt{x^2+x}-x}{x}$.

- At infinity, direct substitution does not work directly, and we need to manipulate the expression.
- Factor out x from the square root and simplify: $= \lim_{x \to \infty} \frac{\sqrt{x^2(1+\frac{1}{x})}-x}{x} = \lim_{x \to \infty} \frac{x\sqrt{1+\frac{1}{x}}-x}{x}$

$= \lim_{x \to \infty}(\sqrt{1+\frac{1}{x}}-1)$

- Direct substitution now yields: as $x \to \infty$, $\frac{1}{x} \to 0$
- $\sqrt{1+0}-1=0$.

1-3. Simplifying Functions Through Factoring for Substitution

Simplifying functions through factoring is a powerful technique used in calculus to make limit evaluations more straightforward, especially when direct substitution results in an indeterminate form like $\frac{0}{0}$.

Factoring allows us to cancel out terms in the numerator and denominator, which may prevent direct evaluation but become resolvable after simplification.

The process generally involves

1. Identifying common factors in the numerator and the denominator.
2. Factoring out these common elements.
3. Simplifying the expression by canceling out the common factors.
4. Substituting the limit point into the simplified function, if now possible.

This approach is particularly useful in preparing a function for limit evaluation using substitution, allowing for clearer insights into the function's behavior near the point of interest.

Formulas:

- Factoring Quadratic Expressions: For a quadratic equation of the form $ax^2 + bx + c$, factorization involves finding two numbers that multiply to ac and add to b.
- Difference of Squares:
- $a^2 - b^2 = (a-b)(a+b)$
- Sum/Difference of Cubes:
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Examples:

1) Evaluate $\lim_{x\to 2}\frac{x^2-4}{x-2}$.

- Factor the numerator using the difference of squares: $\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}$
- Cancel the common factor of $x-2$, simplifying to $x+2$ (for $x\neq 2$): $\lim_{x\to 2}(x+2)=4$
- The limit is 4.

2) Find $\lim_{x\to -1}\frac{x^3+1}{x+1}$.

- Factor the numerator using the sum of cubes: $\frac{x^3+1}{x+1}=\frac{(x+1)(x^2-x+1)}{x+1}$
- Cancel the $x+1$ term: $\lim_{x\to -1}(x^2-x+1)=(-1)^2-(-1)+1=3$
- The limit is 3.

3) Calculate $\lim_{x\to 4}\frac{x^2-16}{x^2-8x+16}$.

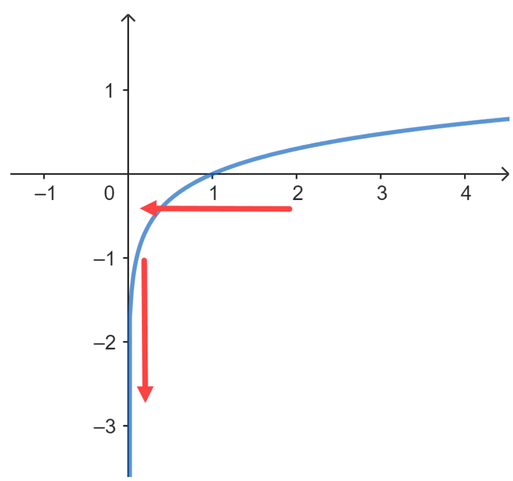
- Both the numerator and denominator can be factored: $\lim_{x\to 4}\frac{(x+4)(x-4)}{(x-4)(x-4)}=\lim_{x\to 4}\frac{x+4}{x-4}=\frac{8}{0}$
- However, since the denominator approaches 0 as x approaches 4, the limit **does not exist due to a division by zero**.

4) Find $\lim_{x \to 0} \frac{1}{x}$

- As x approaches 0, the value of $\frac{1}{x}$ grows without bound.
- Therefore, the limit **does not exist** because it approaches infinity.

5) Find $\lim_{x \to 0^+} \log(x)$

- The natural logarithm of x as x approaches 0 from the right-hand side goes to negative infinity, since the logarithm of values between 0 and 1 is negative and decreases without bound as x approaches 0.
- As $x \to 0^+$ (approaching from the right), the logarithm function $\log(x)$ approaches $-\infty$.
- $\lim_{x \to 0^+} \log(x) = -\infty$



1-4. Limit Estimation Using Calculators (e.g., TI-84+)

Estimating limits using calculators like the TI-84+ can be particularly useful when dealing with complex functions where analytical solutions are difficult to derive or verify.

Calculators can provide numerical approximations to limits by evaluating the function at points increasingly close to the point of interest.

This method is not exact but offers practical insights into the behavior of the function near the target value.

Procedure

- Use the table feature of the calculator.
- Set the table to increment at smaller intervals around the limit point.
- Observe the function values as they converge to a point.

Calculators typically compute these approximations by

1. Allowing you to input a function.

2. Enabling you to set a viewing window that zooms in on the area around the limit point.

3. Providing a "table" feature where you can view outputs of the function at values close to the limit point.

4. Using a "trace" feature to visually determine the behavior as x-values approach the limit point.

- It's important to remember that calculators can encounter rounding and computational limits, which might affect the accuracy of the result, especially for very sensitive limits.

Formulas:

There are No specific formulas for estimating limits using calculators, but the process generally involves:

- Setting the function into the calculator.
- Defining a range around the limit point.
- Observing the output values as the input approaches the limit point.

1) Estimate $\lim_{x\to 0}\frac{\sin x}{x}$ using a TI-84+.

- Enter the function $\frac{\sin x}{x}$ into the "Y=" menu.
- Set the viewing window with x values close to 0 (e.g., -0.1 to 0.1).
- Use the table feature to evaluate the function at $x=-0.01,-0.001,0.001,0.01$.
- Observe that the function values approach 1 as x gets closer to 0 from both sides.

2) Estimate $\lim_{x\to 1}\frac{x^2-1}{x-1}$ using a TI-84+.

- Input $\frac{x^2-1}{x-1}$ into the calculator.
- Set a window around $x=1$, perhaps 0.9 to 1.1.
- Open the table and examine values around $x=1$.
- The values around $x=1$ will indicate that the function approaches 2, which aligns with the algebraic evaluation after factoring and simplifying $(x+1)$.

3) Estimate $\lim_{x\to 2}\frac{x^2-4}{x-2}$ using a calculator's table feature.

- Set up the table feature to increment values of x around 2 (for example, 1.9, 1.99, 1.999 2.001, 2.01, 2.1) and observe the y values.
- The value should approach 4 as x **approaches 2**.

1-5. Graphical Limit Estimation and Visualization

Graphical estimation of limits involves analyzing the behavior of a function as it approaches a certain point from the plot or graph of the function.

This method is particularly useful for visual learners and can provide intuitive insights into the behavior of functions near points of interest, including points of discontinuity or where the function does not have an explicit value.

Procedure

- Plot the graph of the function around the point of interest.
- Observe the behavior of the function as the input values get closer to the target point from both directions.

The key to graphical limit estimation is understanding that:

- If the function approaches a particular y-value as x approaches a certain value from the left and right, then the limit exists at that point.
- If the function approaches different y-values from the left and the right, the limit does not exist at that point (demonstrating a "jump" discontinuity).
- If the function grows without bound (either positively or negatively) as x approaches a certain value, the limit is considered to be infinite.

Using graphing tools, either software like Desmos and GeoGebra or graphing calculators, can aid in this visualization by providing a clear depiction of the function and its limits.

Formulas:

Graphical limit estimation does not involve specific **Formulas:** but focuses on interpreting the behavior of the function's graph:

- Analyze how the y-values change as x-values approach the limit point from both the left ($x \to a^-$) and right ($x \to a^+$).
- Observe any vertical asymptotes where the function may approach infinity.

Examples:

1) Estimate the limit of $f(x)=\frac{1}{x}$ as $x\to 0$ graphically.

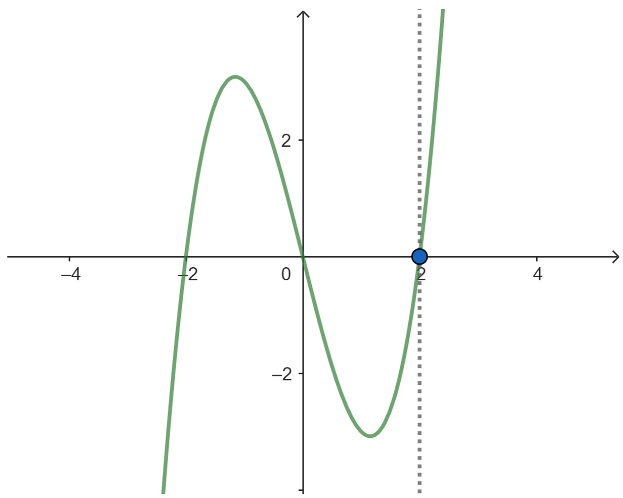
- Plot $f(x)=\frac{1}{x}$.
- Observe that as x approaches 0 from the left, the y-values decrease towards negative infinity. As x approaches 0 from the right, the y-values increase towards positive infinity.
- The behavior indicates that the limit does not exist as $x\to 0$ because the function approaches different values from each direction.

2) Graphically estimate the limit of $f(x)=\frac{x^2-4}{x-2}$ as $x\to 2$.

- Graph the function. Notice the hole at $x=2$ due to the common factor in the numerator and denominator that can be canceled.
- After simplifying the function to $x+2$, re-plot to confirm continuity except for the hole at $x=2$.
- By examining the graph near $x=2$, it can be seen that the function approaches 4.

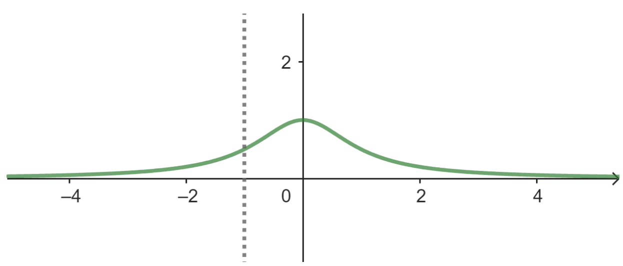
3) Graph the function $f(x)=x^3-4x$ and estimate $\lim_{x\to 2} f(x)$.

- By plotting the graph, you would observe that as x approaches 2, $f(x)$ approaches 0.
- Thus, $\lim_{x\to 2} f(x)=0$.



4) Estimate $\lim_{x\to -1}\frac{1}{x^2+1}$ by graphing.

- The graph of $\frac{1}{x^2+1}$ shows a horizontal asymptote as x approaches -1, and the y value is 0.5.
- So, $\lim_{x\to -1}\frac{1}{x^2+1}=0.5$.



1-6. Algebraic Manipulation and Properties of Limits

Algebraic manipulation in the context of limits involves rearranging and simplifying expressions to make limit calculations more straightforward.

Understanding the properties of limits is essential for effectively applying these techniques, as these properties allow the manipulation of limits in ways that are analogous to standard algebraic operations.

The properties of limits provide a set of rules that help in the computation of limits, ensuring that the limit of a combination of functions can often be determined from the limits of the individual functions, provided those limits exist.

These properties are crucial when dealing with sums, products, quotients, and compositions of functions.

Formulas:

Key properties of limits include:

- Sum of Limits: $\lim_{x \to a}(f(x)+g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- Product of Limits: $\lim_{x \to a}(f(x) \cdot g(x)) = (\lim_{x \to a} f(x)) \cdot (\lim_{x \to a} g(x))$
- Quotient of Limits: $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$
- Limit of a Constant Times a Function: $\lim_{x \to a}(c \cdot f(x)) = c \cdot \lim_{x \to a} f(x)$
- Power of a Limit: $\lim_{x \to a}(f(x))^n = (\lim_{x \to a} f(x))^n$

Examples:

1) Using algebraic manipulation and properties to compute $\lim_{x \to 3}(2x^2 + 5x - 3)$ **.**

- Break the limit into simpler parts using the sum and constant multiple properties:
  $\lim_{x \to 3}(2x^2 + 5x - 3) = 2\lim_{x \to 3} x^2 + 5\lim_{x \to 3} x - \lim_{x \to 3} 3$
- Compute each limit: $2 \cdot 9 + 5 \cdot 3 - 3 = 18 + 15 - 3 = 30$
- The limit is 30.

2) Find $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$ **using algebraic manipulation.**

- Factorize and simplify the expression first: $\frac{x^2 - 1}{x + 1} = \frac{(x - 1)(x + 1)}{x + 1}$
- Simplify (for $x \neq -1$): $x - 1$
- Now, compute the limit: $\lim_{x \to -1}(x - 1) = -1 - 1 = -2$
- The limit is -2.

3) Compute $\lim_{x \to 4} \frac{2x + 1}{3x - 5}$ **.**

- Use the quotient rule, assuming the denominator isn't zero:
- $\lim_{x \to 4} \frac{2x + 1}{3x - 5} = \frac{\lim_{x \to 4}(2x + 1)}{\lim_{x \to 4}(3x - 5)} = \frac{9}{7}$.

4) Evaluate $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$**.**

- Recognize that as x approaches 0, $\sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1, and x becomes very small.
- The product of a very small number and a number that oscillates between -1 and 1 is also very small.
- Therefore, the limit is 0.

5) Evaluate the limit: $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$**.**

- To evaluate the limit $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$, we can use the technique of multiplying by the conjugate to simplify the expression
- $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2}$
- Simplifying the numerator: $= \lim_{x \to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x \to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)}$
- Cancel out the common factor $x-4$: $= \lim_{x \to 4} \frac{1}{\sqrt{x}+2}$
- Substitute $x = 4$: $= \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$
- So, $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} = \frac{1}{4}$.

1-7. Definition and Understanding of One-Sided Limits

One-sided limits are a crucial aspect of calculus that focuses on the behavior of a function as the input approaches a particular point from only one side—either from the left or from the right.

These limits are particularly useful for analyzing functions at points where the behavior is different depending on the direction from which the point is approached.

- **Left-Hand Limit:** The limit of $f(x)$ as x approaches a value a from the left, denoted as $\lim_{x\to a^-} f(x)$.
- **Right-Hand Limit**: The limit of $f(x)$ as x approaches a value a from the right, denoted as $\lim_{x\to a^+} f(x)$.

The distinction between left-hand and right-hand limits is essential for understanding discontinuities and determining whether the overall limit exists at a point. The **overall limit** $\lim_{x\to a} f(x)$ exists only if both the left-hand limit and the right-hand limit exist and are equal.

Two-sided limits refer to the value a function approaches as the variable gets close to a particular point from both directions (left and right). If the function approaches the same value from both sides, the two-sided limit exists. This situation is mathematically denoted as $\lim_{x\to c} f(x) = L$.

One-sided limits are concerned with the behavior of functions as the variable approaches a point from one direction only, either from the left ($x \to c^-$) or from the right ($x \to c^+$): from the right (indicated as $x \to c^+$), the limit is written as $\lim_{x\to c^+} f(x) = L$, and from the left (indicated as $x \to c^-$), it's written as $\lim_{x\to c^-} f(x) = L$.

- The **existence of limits** is determined by the behavior of one-sided limits. A two-sided limit exists only if both one-sided limits are equal; otherwise, the limit does not exist (DNE).
- **Infinite limits** occur when the function grows without bound as the variable approaches a particular point. These are denoted as $+\infty$ or $-\infty$, depending on the direction of the growth.

Formulas:

- Left-Hand Limit: $\lim_{x \to a^-} f(x) = L$ means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if x is within δ of a from the left ($a - \delta < x < a$), then $|f(x) - L| < \epsilon$.
- Right-Hand Limit: $\lim_{x \to a^+} f(x) = L$ follows a similar definition, but x approaches a from the right ($a < x < a + \delta$).

Examples:

1) Compute the left-hand and right-hand limits of $f(x) = \frac{x}{|x|}$ as $x \to 0$.

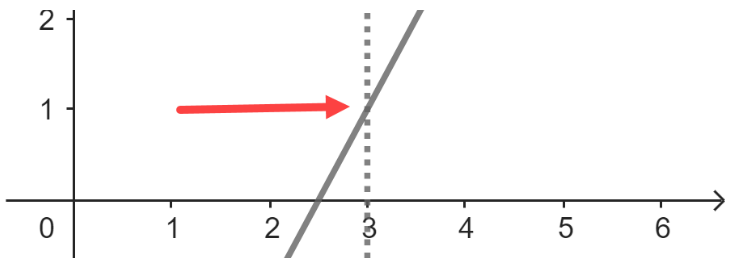
- Left-Hand Limit: $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = -1$
- Right-Hand Limit: $\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = 1$
- Since the left-hand and right-hand limits are **not equal**, the overall limit $\lim_{x \to 0} \frac{x}{|x|}$ **does not exist** (DNE).

2) Find $\lim_{x \to 2^-} (x^2 - 4)$ and $\lim_{x \to 2^+} (x^2 - 4)$.

- Left-Hand Limit: $\lim_{x \to 2^-} (x^2 - 4) = 4 - 4 = 0$
- Right-Hand Limit: $\lim_{x \to 2^+} (x^2 - 4) = 4 - 4 = 0$
- Both limits are equal, so $\lim_{x \to 2} (x^2 - 4) = 0$.

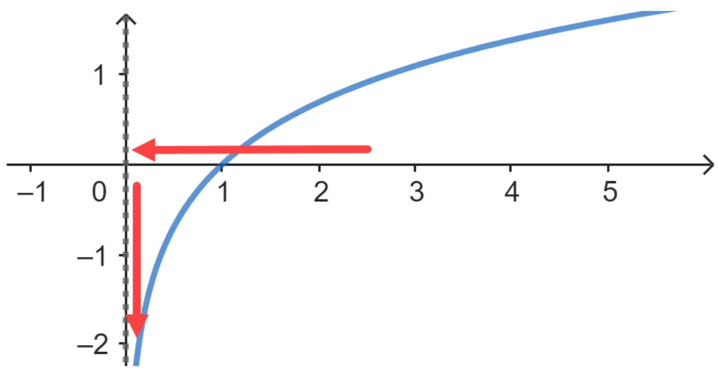
3) Evaluate the left-hand limit $\lim_{x \to 3^-} (2x - 5)$ **.**

- Direct substitution since it's a continuous function: $\lim_{x \to 3^-} (2x - 5) = 2(3) - 5 = 1$.



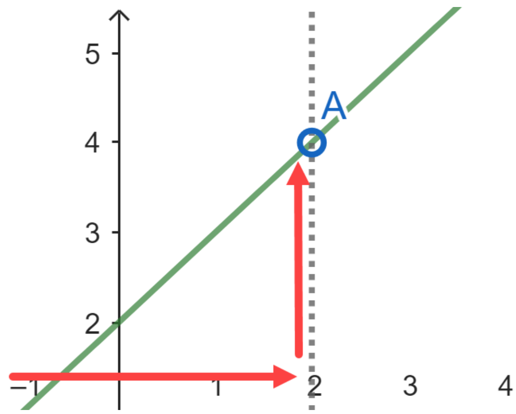
4) Evaluate the right-hand limit $\lim_{x \to 0^+} \ln(x)$ **.**

- As x approaches 0 from the right, $\ln(x)$ goes to negative infinity. So, the right-hand limit is $-\infty$.



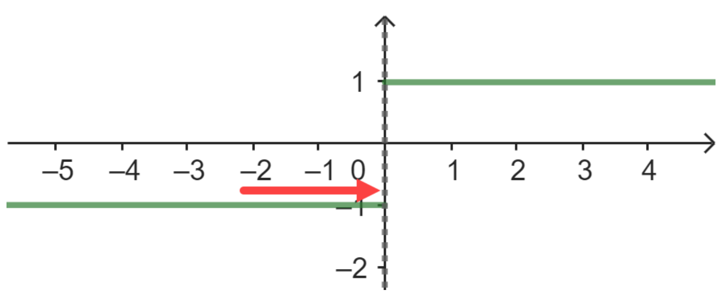
5) Compute the left-hand limit $\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2}$.

- Factor and cancel to find $\lim_{x \to 2^-} \frac{(x+2)(x-2)}{x-2}$. After canceling, the left-hand limit is $2+2=4$.



6) Compute the left-hand limit $\lim_{x \to 0^-} \frac{|x|}{x}$.

- As x approaches 0 from the left, $|x| = (-x)$. Thus, $\lim_{x \to 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$.



1-8. Understanding and Applying the Concepts of Limits in Calculus

Limits are a fundamental concept in calculus, forming the basis for defining derivatives and integrals.

Understanding limits involves grasping how a function behaves as its inputs approach a specific value, regardless of whether the function is actually defined at that value. This is essential for analyzing functions that are not continuous or have points of discontinuity.

The concept of a limit helps to:

- Understand instantaneous rates of change (derivatives).
- Handle functions that are undefined or indeterminate at certain points.
- Determine the behavior of sequences and series.
- Deal with infinitesimally small quantities.

Limits can be approached from a graphical perspective, using numerical approximations, or through symbolic manipulations. Each method offers unique insights and is useful in different contexts.

Formulas:

- **Basic Limit**: $\lim_{x \to a} f(x) = L$ signifies that as x approaches a, $f(x)$ approaches L.
- **Squeeze Theorem**: If $g(x) \le f(x) \le h(x)$ for all x near a (except possibly at a) and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.
- **L'Hôpital's Rule**: If $\lim_{x \to a} \frac{f(x)}{g(x)}$ results in an indeterminate form like $0/0$ or ∞/∞, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limits of the derivatives exist.

Examples:

1) Apply the Squeeze Theorem to find $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$ **.**

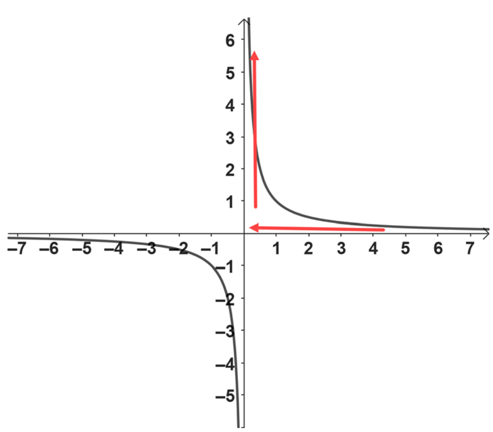
- Use the inequalities $-1 \le \sin(\frac{1}{x}) \le 1: \Rightarrow -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$
- Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$

2) Use L'Hôpital's Rule to evaluate $\lim_{x\to 0}\frac{\sin x}{x}$ **.**

- Initially, $\frac{\sin x}{x}$ at $x = 0$ gives $\frac{0}{0}$, an indeterminate form.
- Use L'Hôpital's Rule $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$: $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \cos(0) = 1$

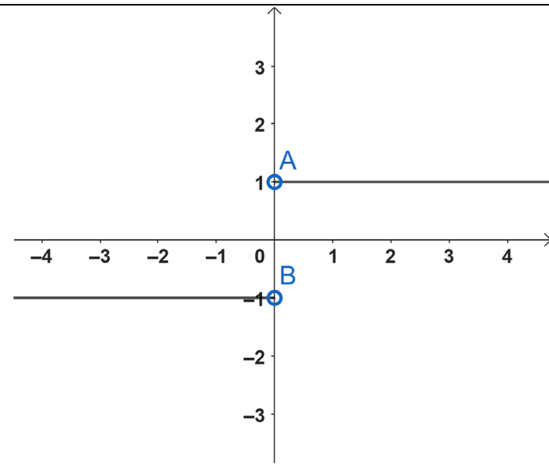
3) If $f(x) = \frac{1}{x}$**, what is** $\lim_{x\to 0^+} f(x)$ **?**

- $\lim_{x\to 0^+} f(x)$ **is** $+\infty$ because as x approaches 0 from the positive side, the function's value increases without bound.



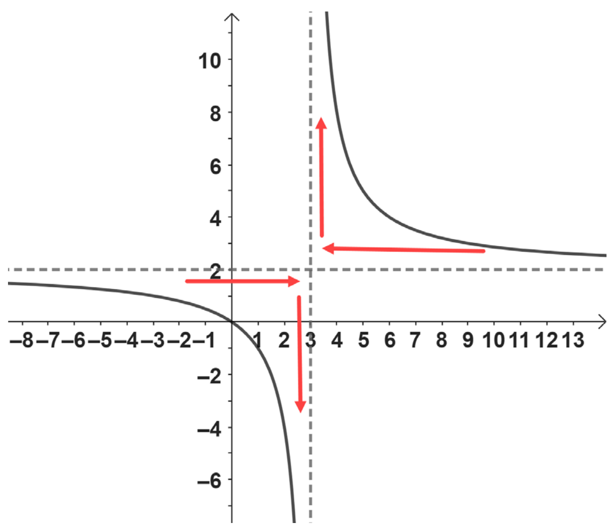
4) Given $f(x) = \frac{|x|}{x}$, does $\lim_{x \to 0} f(x)$ exist?

- **No**, $\lim_{x \to 0} f(x)$ does not exist (**DNE**) because
- the left-hand limit ($x \to 0^-$) is -1
- the right-hand limit ($x \to 0^+$) is 1
- They are **not equal**.



5) If $f(x) = \frac{2x}{x-3}$, determine $\lim_{x \to 3^+} f(x)$ and $\lim_{x \to 3^-} f(x)$.

- $\lim_{x \to 3^+} f(x)$ is $+\infty$ and $\lim_{x \to 3^-} f(x)$ is $-\infty$, because as x approaches 3 from the right, the function goes to positive infinity, and from the left, it goes to negative infinity.



1-9. Formal Limit Definition Based on Closeness

The epsilon-delta definition of a limit formalizes the concept of limits in calculus using precise mathematical language.

This definition is used to prove whether a limit exists at a point according to how closely $f(x)$ approaches a specific value as x approaches a certain point.

The key idea is that for every small number ϵ (representing the distance around the limit L), there exists a corresponding small number δ (representing the distance around the point a) such that whenever $0<|x-a|<\delta$, it follows that $|f(x)-L|<\epsilon$.

Formulas:

- $\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$,
  there exists a $\delta > 0$ such that if $0<|x-a|<\delta$, then $|f(x)-L|<\epsilon$.

Understand Concepts by Game:

Imagine you're playing a game where your goal is to get as close as possible to a target without actually touching it. In math, when we say a function approaches a limit as gets close to a certain value, we're talking about something similar. The function's values get closer and closer to some number (the limit) as x approaches a specific point.

- **Definition**: The Formal Rules of the Game
- **Function** f: This is like the path you're walking on in our game.
- **Limit** L: This is the target spot you're trying to get close to.
- x **approaches** a: You're walking towards a specific point a on your path.
- **For every** $\epsilon > 0$: Think of ϵ as a challenge level in the game, where you're asked to get within a certain distance of the target. No matter how small this distance is (as long as it's greater than zero), you need to show you can meet the challenge.
- **There exists a** $\delta > 0$: This is your strategy for the challenge. Based on how tough the challenge is (ϵ), you decide how close you need to get to the point a (this closeness is δ) to ensure you're within the target distance (ϵ) of the limit L.
- **If** $0<|x-a|<\delta$: This means you're within your decided closeness to the point a, but not exactly at a.

- **Then** $|f(x)-L|<\epsilon$: If you're within that closeness, then you've succeeded in the challenge; the function's value is within the target distance of the limit L.

Putting It Into Practice (**hint**: find matching positive ϵ & δ by $|f(x)-L|<\epsilon$ = $|x-a|<\delta$)

Examples:

1) Prove that $\lim_{x\to 3} x^2 = 9$ **using the epsilon-delta definition.**

- Choose $\epsilon > 0$. We need to find $\delta > 0$ such that if $0 < |x-3| < \delta$, then $|x^2-9| < \epsilon$.
- Notice that $|x^2-9| = |x+3||x-3|$.
- Assume $|x-3| < 1$ (which implies $2 < x < 4$ and thus $5 < x+3 < 7$).
- Now, $|x^2-9| = |x+3||x-3| < 7|x-3|$.
- Set $\delta = \min\left(1, \frac{\epsilon}{7}\right)$. Then $|x^2-9| < 7|x-3| < 7\delta \le \epsilon$.

2) Determine δ **for** $\lim_{x\to 2}(3x-1) = 5$ **given** $\epsilon = 0.1$**.**

- We have $|3x-5| < \epsilon$ whenever $0 < |x-2| < \delta$.
- Simplify $|3x-5| = 3|x-2|$. We want $3|x-2| < 0.1$, so $|x-2| < \frac{0.1}{3}$.
- Therefore, $\delta = \frac{0.1}{3}$.

3) Let's say you're proving that as x gets close to 2, the function $f(x)=x^2$ gets close to 4. Here's how you'd use the epsilon-delta definition:

- **Challenge Level** (ϵ): Someone challenges you to show that you can get the function's value within, say, 0.1 units of 4.
- **Strategy** (δ): You figure out that if you choose x values within 0.05 units of 2, then $f(x)$ will be within 0.1 units of 4.
- **Verification**: You then check mathematically that whenever x is within 0.05 units of 2 (but not exactly 2), $f(x)$ (or x^2 in this case) is indeed within 0.1 units of 4. If this works out, you've met the challenge according to the game's rules.

In essence, the epsilon-delta definition is a formal way to say, "No matter how close you want to get to the target, I can find a way to get the function's values within that closeness by picking x values close enough to a certain point." It's about proving that you can always meet the challenge of getting close to the limit.

4) Prove that $\lim_{x \to 3} x = 3$ using the epsilon-delta definition of a limit.

- For every $\epsilon > 0$, we need to find a $\delta > 0$ such that if $0 < |x-3| < \delta$, then $|x-3| < \epsilon$.
- Choosing $\delta = \epsilon$ works since if $0 < |x-3| < \delta$, then $|x-3| < \epsilon$.

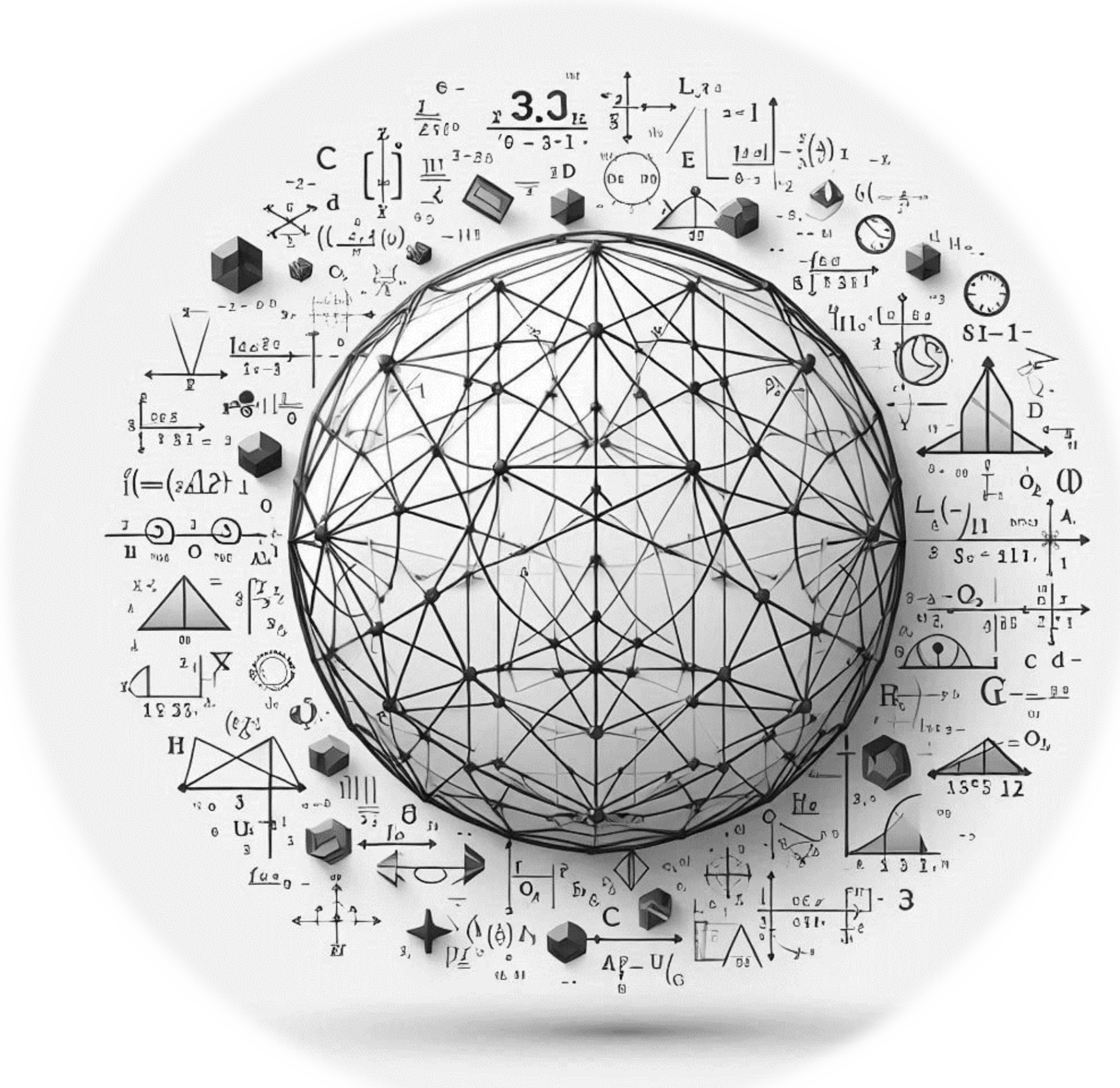
5) Using the epsilon-delta definition, prove that $\lim_{x \to 2}(3x-4) = 2$.

- For every $\epsilon > 0$, choose $\delta = \epsilon / 3$.
- If $0 < |x-2| < \delta$, then $|3x-4-2| = 3|x-2| < 3\delta = \epsilon$, hence the limit is proven.

6) Use the epsilon-delta definition to prove that $\lim_{x\to 0}\frac{1}{1+x}=1$.

- For every $\epsilon>0$, choose $\delta=\min\left\{1,\frac{\epsilon}{1+\epsilon}\right\}$. If $0<|x|<\delta$, then $\left|\frac{1}{1+x}-1\right|=\left|\frac{-x}{1+x}\right|=\frac{|x|}{|1+x|}$.
- Since $|x|<\delta\le 1$, $|1+x|>1$.
- Thus, $\frac{|x|}{|1+x|}<|x|<\delta\le\epsilon$, establishing the proof.
- Using epsilon-delta, prove that $\lim_{x\to a}c=c$ for any constant c and real number a.
- For every $\epsilon>0$, let $\delta=\epsilon$.
- Then if $0<|x-a|<\delta$, $|f(x)-L|=|c-c|=0<\epsilon$.
- Since 0 is always less than ϵ, the limit is proven.

Chapter 11. Infinite Sequences and Series (AP BC Only)



11-1. Defining Convergent and Divergent Infinite Series

An **infinite series** is the sum of the terms of an infinite sequence. It can be written in the form:

$$\sum_{n=1}^{\infty} a_n$$

where a_n represents the terms of the series.

Convergence of an infinite series occurs when the sum of its terms approaches a finite limit as the number of terms increases. Conversely, a series is divergent if the sum does not approach a finite limit.

Definitions:

1. **Convergent Series:** An infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence of partial sums S_N approaches a finite limit L as N approaches infinity:

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = L$$

If this limit exists and is finite, the series converges to L.

2. **Divergent Series:** An infinite series $\sum_{n=1}^{\infty} a_n$ is divergent if the sequence of partial sums S_N does not approach a finite limit as N approaches infinity:

$$\lim_{N\to\infty} S_N = \pm\infty \text{ or does not exist}$$

Tests for Convergence and Divergence

Several tests can determine whether a series is convergent or divergent.

1. **n-th Term Test for Divergence**: If the limit of a_n as n approaches infinity is not zero, then the series $\sum_{n=1}^{\infty} a_n$ is divergent: $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

2. **Geometric Series Test**: A geometric series $\sum_{n=0}^{\infty} ar^n$ converges if $|r|<1$ and diverges if $|r|\geq 1$. The sum of a convergent geometric series is: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r|<1$

3. **p-Series Test:** A p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p>1$ and diverges if $p\leq 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p>1$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p\leq 1$$

4. **Comparison Test:** Compare the given series with a known series:

- If $0\leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $a_n \geq b_n \geq 0$ for all n and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

5. **Integral Test:** If $a_n = f(n)$ where $f(x)$ is a positive, continuous, and decreasing function for $x\geq 1$, then the series $\sum a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

Examples:

1) Geometric Series

- Determine if the series $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ is convergent or divergent.

- This is a geometric series with $a=1$ and $r=\frac{1}{3}$. Since $|r|=\frac{1}{3}<1$, the series converges.
- The sum is:

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}$$

2) p-Series

- Determine if the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent or divergent.

- This is a p-series with $p=2$.
- Since $p>1$, the series converges.

3) Comparison Test

- Determine if the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ is convergent or divergent.

- Compare $\frac{1}{n^2+1}$ with $\frac{1}{n^2}$.
- Since $\frac{1}{n^2+1}<\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series with $p=2$), by the comparison test, $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ also converges.

4) Integral Test

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ is convergent or divergent.

- Let $f(x) = \frac{1}{x\ln(x)}$.
- Evaluate the improper integral: $\int_{2}^{\infty} \frac{1}{x\ln x} dx$
- Use substitution $u = \ln x$, $du = \frac{1}{x} dx$: $\int_{2}^{\infty} \frac{1}{x\ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du = \left[\ln u\right]_{\ln 2}^{\infty}$
- The integral diverges because $\ln u$ approaches infinity as approaches infinity.
- Hence, the series $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ diverges.

11-2. Working with Geometric Series

A **geometric series** is a series where each term is a constant multiple (called the common ratio) of the previous term. Geometric series can be either finite or infinite, and their properties depend heavily on the value of the common ratio.

A geometric series can be expressed as:

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

where a is the first term and r is the common ratio.

Formulas:

1. **Sum of an Infinite Geometric Series:**

An infinite geometric series converges if the absolute value of the common ratio is less than 1 (|r| < 1). The sum S of an infinite geometric series is given by:

$$S = \frac{a}{1-r} \quad \text{for} \, |r| < 1$$

If $|r| \geq 1$, the series diverges.

2. **Sum of a Finite Geometric Series:**

The sum S_n of the first n terms of a finite geometric series is given by:

$$S_n = a\frac{1-r^n}{1-r} \quad \text{for} \, r \neq 1$$

Examples:

1) Sum of an Infinite Geometric Series

- Find the sum of the infinite geometric series $\sum_{n=0}^{\infty} 4\left(\frac{1}{3}\right)^n$.

- Identify the first term a and the common ratio r : $a = 4, \quad r = \frac{1}{3}$

- Since $|r|<1$, the series converges. The sum is: $S=\frac{a}{1-r}=\frac{4}{1-\frac{1}{3}}=\frac{4}{\frac{2}{3}}=4\cdot\frac{3}{2}=6$

2) Sum of a Finite Geometric Series

- Find the sum of the first 5 terms of the geometric series $3+6+12+24+\cdots$.

- Identify the first term a and the common ratio r : $a=3, \quad r=2$
- Use the formula for the sum of the first n terms: $S_5=a\frac{1-r^n}{1-r}=3\frac{1-2^5}{1-2}=93$

3) Convergence of an Infinite Geometric Series

- Determine if the series $\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n$ converges or diverges.

- Identify the first term a and the common ratio r : $a=1, \quad r=-\frac{2}{5}$
- Since $|r|<1$, the series converges. The sum is: $S=\frac{a}{1-r}=\frac{1}{1-\left(-\frac{2}{5}\right)}=\frac{5}{7}$

4) Divergence of an Infinite Geometric Series

- Determine if the series $\sum_{n=0}^{\infty}5(2)^n$ converges or diverges.

- Identify the first term a and the common ratio r : $a=5, \quad r=2$
- Since $|r|\geq 1$, the series diverges.

11-3. The n th Term Test for Divergence

The n th Term **Test for Divergence** is a simple yet powerful test used to determine whether an infinite series diverges. This test is based on the behavior of the terms of the series as n approaches infinity.

If the limit of the n th term of the series does not approach zero, then the series diverges. However, if the limit of the n th term does approach zero, the test is inconclusive, and other tests must be applied to determine convergence or divergence.

Formulas:

n th Term Test for Divergence:

For a series $\sum_{n=1}^{\infty} a_n$,

- if: $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist
- then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Examples:

1) Divergent Series

- Determine if the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges.

- Evaluate the limit of the n th term as n approaches infinity: $a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1+0 = 1 \neq 0$$

- Since $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges.

2) Inconclusive Test

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges using the n th Term Test for Divergence.

- Evaluate the limit of the n th term as n approaches infinity: $a_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

- Since $\lim_{n\to\infty} a_n = 0$, the n th Term Test for Divergence is inconclusive.
- This series is known as the harmonic series, and it diverges, but we would need to use a different test (such as the integral test) to establish this.

3) Convergent Series

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges using the n th Term Test for Divergence.

- Evaluate the limit of the n th term as n approaches infinity: $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

- Since $\lim_{n\to\infty} a_n = 0$, the n th Term Test for Divergence is inconclusive.
- To determine convergence, we can use the p-series test. Since this is a p-series with $p = 2 > 1$, the series converges.

4) Series with Trigonometric Terms

- Determine if the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ converges or diverges using the n th Term Test for Divergence.

- Evaluate the limit of the n th term as n approaches infinity: $a_n = \sin\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$$

- Since $\lim_{n\to\infty} a_n = 0$, the n th Term Test for Divergence is inconclusive.
- This means the test does not tell us whether the series converges or diverges.
- Therefore, we cannot determine the convergence or divergence of the series using this test alone. Other tests or methods would be needed to further analyze the series.

11-4. Integral Test for Convergence

The **Integral Test for Convergence** is a powerful tool for determining the convergence or divergence of an infinite series. This test is particularly useful when dealing with series whose terms are generated by a positive, continuous, and decreasing function.

The Integral Test connects the convergence of an infinite series $\sum_{n=1}^{\infty} a_n$ with the convergence of an improper integral $\int_1^{\infty} f(x)dx$, where $f(x) = a_n$ for $x = n$.

Theorem (Integral Test):

Let $f(x)$ be a positive, continuous, and decreasing function for $x \geq 1$, and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges:

$$\sum_{n=1}^{\infty} a_n \text{ converges } \Rightarrow \int_1^{\infty} f(x)dx \text{ converges}$$

If the integral diverges, then the series also diverges.

Examples:

1) Convergence of a p-Series

- Determine if the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges or diverges using the Integral Test.

- The given series is a p-series with $p=2$. We apply the Integral Test by evaluating the corresponding improper integral: $\int_1^{\infty}\frac{1}{x^2}dx$
- Find the antiderivative: $\int\frac{1}{x^2}dx=\int x^{-2}dx=-x^{-1}=-\frac{1}{x}$
- Evaluate the improper integral: $\int_1^{\infty}\frac{1}{x^2}dx=\lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b=\lim_{b\to\infty}\left(-\frac{1}{b}+1\right)=1$
- Since the integral converges to 1, the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

2) Divergence of the Harmonic Series

- Determine if the series $\sum_{n=1}^{\infty}\frac{1}{n}$ converges or diverges using the Integral Test.

- The given series is the harmonic series. We apply the Integral Test by evaluating the corresponding improper integral: $\int_1^{\infty}\frac{1}{x}dx$
- Find the antiderivative: $\int\frac{1}{x}dx=\ln|x|=\ln x$
- Evaluate the improper integral: $\int_1^{\infty}\frac{1}{x}dx=\lim_{b\to\infty}\left[\ln x\right]_1^b=\lim_{b\to\infty}\left(\ln b-\ln 1\right)=\lim_{b\to\infty}\ln b=\infty$
- Since the integral diverges, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

3) Series with a Non-Integer Power

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges or diverges using the Integral Test.

- The given series is a p-series with $p = \frac{3}{2}$. We apply the Integral Test by evaluating the corresponding improper integral: $\int_1^{\infty} \frac{1}{x^{3/2}} dx$
- Find the antiderivative: $\int \frac{1}{x^{3/2}} dx = \int x^{-3/2} dx = \int x^{-3/2} dx = \frac{x^{-1/2}}{-1/2} = -2x^{-1/2} = -\frac{2}{\sqrt{x}}$
- Evaluate the improper integral: $\int_1^{\infty} \frac{1}{x^{3/2}} dx = \lim_{b \to \infty} \left[-\frac{2}{\sqrt{x}} \right]_1^b = \lim_{b \to \infty} \left(-\frac{2}{\sqrt{b}} + 2 \right) = 2$
- Since the integral converges to 2, the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

4) Series with a Logarithmic Term

- Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges using the Integral Test.

- We apply the Integral Test by evaluating the corresponding improper integral: $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$
- Use substitution $u = \ln x$, $du = \frac{1}{x} dx$: $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\infty} \frac{1}{u^2} du$
- Find the antiderivative: $\int \frac{1}{u^2} du = -\frac{1}{u}$
- Evaluate the improper integral: $\int_{\ln 2}^{\infty} \frac{1}{u^2} du = \lim_{b \to \infty} \left[-\frac{1}{u} \right]_{\ln 2}^b = \lim_{b \to \infty} \left(-\frac{1}{b} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}$
- Since the integral converges to $\frac{1}{\ln 2}$, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

11-5. Harmonic Series and p-Series

The harmonic series and p-series are specific types of infinite series that have important implications in calculus. Understanding these series helps in determining convergence and divergence using various tests.

Harmonic Series:

The harmonic series is the series of the reciprocals of the natural numbers:

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

The harmonic series is divergent. This can be shown using the Integral Test or by comparing the harmonic series to a known divergent series.

p-Series:

A p-series is a series of the form: $\sum_{n=1}^{\infty}\frac{1}{n^p}$ where p is a positive constant.

- The p-series converges If $p>1$.
- The p-series diverges if $p\le 1$..

Theorems and Tests

1. p-Series Test:

- If p > 1 , then $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.
- If $p\le 1$, then $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

2. Integral Test:

- To apply the Integral Test to the harmonic series or a p-series, evaluate the corresponding improper integral. For a p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$, consider the integral: $\int_1^{\infty}\frac{1}{x^p}dx$

- If $p > 1$, the integral converges, and so does the series.
- If $p \leq 1$, the integral diverges, and so does the series.

Examples:

1) Divergence of the Harmonic Series

- Determine if the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

- Apply the Integral Test by evaluating the corresponding improper integral: $\int_1^{\infty} \frac{1}{x} dx$
- Find the antiderivative: $\int \frac{1}{x} dx = \ln x$
- Evaluate the improper integral: $\int_1^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} [\ln x]_1^b = \lim_{b \to \infty} (\ln b - \ln 1) = \lim_{b \to \infty} \ln b = \infty$
- Since the integral diverges, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2) Convergence of a p-Series with $p = 2$

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

- Apply the p-Series Test with $p = 2$: Since $p > 1$, the series converges.
- To confirm, apply the Integral Test by evaluating the corresponding improper integral: $\int_1^{\infty} \frac{1}{x^2} dx$
- Find the antiderivative: $\int \frac{1}{x^2} dx = -\frac{1}{x}$
- Evaluate the improper integral: $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{b \to \infty} \left[-\frac{1}{x} \right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{b} + 1 \right) = 1$
- Since the integral converges to 1, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

3) Divergence of a p-Series with $p=1$

Determine if the series $\sum_{n=1}^{\infty}\frac{1}{n^{1}.1}$ converges or diverges.

- Apply the p-Series Test with $p=1.1$: Since $p>1$, the series converges.
- To confirm, apply the Integral Test by evaluating the corresponding improper integral: $\int_{1}^{\infty}\frac{1}{x^{1.1}}dx$
- Find the antiderivative: $\int\frac{1}{x^{1.1}}dx=\frac{x^{-0.1}}{-0.1}=-\frac{1}{0.1}x^{-0.1}=-10x^{-0.1}$
- Evaluate the improper integral: $\int_{1}^{\infty}\frac{1}{x^{1.1}}dx=\lim_{b\to\infty}\left[-10x^{-0.1}\right]_{1}^{b}=\lim_{b\to\infty}\left(-10b^{-0.1}+10\right)=10$
- Since the integral converges to 10, the series $\sum_{n=1}^{\infty}\frac{1}{n^{1.1}}$ converges.

11-6. Comparison Tests for Convergence

Comparison tests are powerful tools for determining the convergence or divergence of an infinite series by comparing it to a series whose convergence properties are already known. There are two main comparison tests: the Direct Comparison Test and the Limit Comparison Test.

1. Direct Comparison Test:

This test involves directly comparing the terms of two series.

- Theorem: Let $\sum a_n$ and $\sum b_n$ be two series with positive terms.
- If $0 \le a_n \le b_n$ for all $n \ge N$ (for some positive integer N), and $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $0 \le b_n \le a_n$ for all $n \ge N$, and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

2. Limit Comparison Test:

This test involves taking the limit of the ratio of the terms of two series.

- Theorem: Let $\sum a_n$ and $\sum b_n$ be two series with positive terms. Suppose:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a positive finite constant.

- If $\sum b_n$ converges, then $\sum a_n$ also converges.
- If $\sum b_n$ diverges, then $\sum a_n$ also diverges.

Examples:

1) Direct Comparison Test

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.

- Compare $\frac{1}{n^2+1}$ with $\frac{1}{n^2}$. Since $n^2 \le n^2+1$, we have:

$$\frac{1}{n^2+1} \le \frac{1}{n^2}$$

- The series $\sum \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$.
- By the Direct Comparison Test, since $\sum \frac{1}{n^2}$ converges and $0 \le \frac{1}{n^2+1} \le \frac{1}{n^2}$, the series $\sum \frac{1}{n^2+1}$ also converges.

2) Limit Comparison Test

- Determine if the series $\sum_{n=1}^{\infty} \frac{5n+1}{n^3+2n}$ converges or diverges.

- Compare $\frac{5n+1}{n^3+2n}$ with $\frac{5n}{n^3} = \frac{5}{n^2}$.
- Calculate the limit: $\lim_{n\to\infty} \frac{\frac{5n+1}{n^3+2n}}{\frac{5}{n^2}} = \lim_{n\to\infty} \frac{(5n+1)n^2}{5(n^3+2n)} = \lim_{n\to\infty} \frac{5n^3+n^2}{5n^3+10n} = \lim_{n\to\infty} \frac{5+\frac{1}{n}}{5+\frac{10}{n^2}} = \frac{5+0}{5+0} = 1$
- Since the limit is a positive finite constant, and $\sum \frac{5}{n^2}$ is a convergent p-series with $p = 2 > 1$, by the Limit Comparison Test, the series $\sum \frac{5n+1}{n^3+2n}$ also converges.

3) Direct Comparison Test with Divergence

- Determine if the series $\sum_{n=1}^{\infty}\frac{2n}{n^2-1}$ converges or diverges.

- Compare $\frac{2n}{n^2-1}$ with $\frac{2n}{n^2}$.
- Since $n^2-1\le n^2$ for all $n\ge 1$, we have: $\frac{2n}{n^2-1}\ge\frac{2n}{n^2}=\frac{2}{n}$
- The series $\sum\frac{2}{n}$ is a divergent harmonic series.
- By the Direct Comparison Test, since $\sum\frac{2}{n}$ diverges and $\frac{2}{n}\le\frac{2n}{n^2-1}$, the series $\sum\frac{2n}{n^2-1}$ also diverges.

4) Limit Comparison Test with Divergence

- Determine if the series $\sum_{n=1}^{\infty}\frac{n+\ln(n)}{n^2}$ converges or diverges.

- Compare $\frac{n+\ln(n)}{n^2}$ with $\frac{n}{n^2}=\frac{1}{n}$.
- Calculate the limit: $\lim_{n\to\infty}\frac{\frac{n+\ln(n)}{n^2}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{(n+\ln n)n}{n^2}=\lim_{n\to\infty}\frac{n+\ln n}{n}=\lim_{n\to\infty}\left(1+\frac{\ln n}{n}\right)=1+0=1$
- Since the limit is a positive finite constant, and $\sum\frac{1}{n}$ is a divergent harmonic series, by the Limit Comparison Test, the series $\sum\frac{n+\ln(n)}{n^2}$ also diverges.

11-7. Alternating Series Test for Convergence

An alternating series is a series in which the terms alternate in sign. The Alternating Series Test, also known as the **Leibniz Test,** is used to determine the convergence of such series.

A typical alternating series can be written in the form:

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n}a_n$$

where a_n are positive terms.

Theorem (Alternating Series Test)

The Alternating Series Test states that an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges if the following two conditions are met:

1. The terms a_n are decreasing: $a_{n+1} \le a_n$ for all n .

2. The terms a_n approach zero: $\lim_{n\to\infty} a_n = 0$.

Examples:

1) Alternating Harmonic Series

- Determine if the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges.

- The given series is: $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$
- Check if the terms $a_n = \frac{1}{n}$ are decreasing: $a_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = a_n$ (This is true for all n.)
- Check if the terms a_n approach zero: $\lim_{n\to\infty}\frac{1}{n} = 0$
- Since both conditions of the Alternating Series Test are met, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges.

2) Alternating Series with Factorial Terms

- Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ converges.

- The given series is: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$
- Check if the terms $a_n = \frac{1}{n!}$ are decreasing: $a_{n+1} = \frac{1}{(n+1)!} \leq \frac{1}{n!} = a_n$ (This is true for all n.)
- Check if the terms a_n approach zero: $\lim_{n\to\infty} \frac{1}{n!} = 0$
- Since both conditions of the Alternating Series Test are met, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ converges.

3) Series Not Approaching Zero

Determine if the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$ converges.

- The given series is: $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$
- Check if the terms $a_n = \frac{n}{n+1}$ are decreasing: $a_{n+1} = \frac{n+1}{n+2} < \frac{n}{n+1} = a_n$
- This is true for all n.
- Check if the terms a_n approach zero: $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$
- Since the terms do not approach zero, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$ diverges.

11-8. Ratio Test for Convergence

The **Ratio Test** is a useful method for determining the convergence or divergence of an infinite series, particularly when the terms of the series involve factorials, exponentials, or other expressions where successive terms can be easily compared.

Theorem (Ratio Test):

Given a series $\sum_{n=1}^{\infty} a_n$, let: $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

- If $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $L > 1$ or $L = \infty$, the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $L = 1$, the test is inconclusive, and the series may converge or diverge.

Examples:

1) Convergence of a Series with Factorials

- Determine if the series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ converges or diverges.

- Apply the Ratio Test: $a_n = \frac{n!}{2^n}$
- Calculate the ratio of successive terms: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!}\right| = \left|\frac{(n+1)\cdot n!}{2\cdot 2^n} \cdot \frac{2^n}{n!}\right| = \left|\frac{n+1}{2}\right|$
- Evaluate the limit: $L = \lim_{n\to\infty}\left|\frac{n+1}{2}\right| = \lim_{n\to\infty}\frac{n+1}{2} = \infty$
- Since $L = \infty$, the series $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ diverges.

2) Convergence of a Series with Exponential Terms

- Determine if the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges or diverges.

- Apply the Ratio Test: $a_n = \frac{3^n}{n!}$
- Calculate the ratio of successive terms: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right| = \left|\frac{3 \cdot 3^n}{(n+1) \cdot 3^n} \cdot \frac{n!}{n!}\right| = \left|\frac{3}{n+1}\right|$
- Evaluate the limit: $L = \lim_{n\to\infty}\left|\frac{3}{n+1}\right| = \lim_{n\to\infty}\frac{3}{n+1} = 0$
- Since $L = 0 < 1$, the series $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ converges absolutely.

3) Inconclusive Ratio Test

- Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges using the Ratio Test.

- Apply the Ratio Test: $a_n = \frac{1}{n}$
- Calculate the ratio of successive terms: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1/(n+1)}{1/n}\right| = \left|\frac{n}{n+1}\right| = 1 - \frac{1}{n+1}$
- Evaluate the limit: $L = \lim_{n\to\infty}\left|1 - \frac{1}{n+1}\right| = 1$
- Since $L = 1$, the Ratio Test is inconclusive.
- The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known to diverge, but we would need a different test (e.g., the Integral Test) to confirm this.

11-9. Determining Absolute or Conditional Convergence

When analyzing the convergence of series, it is important to distinguish between absolute convergence and conditional convergence.

1. **Absolute Convergence**: A series $\sum a_n$ is said to converge absolutely if the series of absolute values $\sum |a_n|$ converges. Absolute convergence implies convergence.

2. **Conditional Convergence**: A series $\sum a_n$ converges conditionally if it converges but does not converge absolutely. This means $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Tests and Definitions

1. Absolute Convergence Test:

- If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.

2. Conditional Convergence:

- If $\sum a_n$ converges but $\sum |a_n|$ diverges, then $\sum a_n$ converges conditionally.

Examples:

1) Absolute Convergence of a Series with Alternating Terms

- Determine if the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges absolutely or conditionally.

- To check for absolute convergence, consider the series of absolute values: $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$
- This is a p-series with $p = 2 > 1$, so it converges.
- Since $\sum |a_n|$ converges, the original series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges absolutely.

2) Conditional Convergence of the Alternating Harmonic Series

- Determine if the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges absolutely or conditionally.

- To check for absolute convergence, consider the series of absolute values: $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$
- This is the harmonic series, which diverges. Since $\sum |a_n|$ diverges, the original series does not converge absolutely.
- Next, check for conditional convergence using the Alternating Series Test:
  The terms $\frac{1}{n}$ are positive, decreasing, and approach zero as $n \to \infty$: $\lim_{n\to\infty}\frac{1}{n} = 0$
- Since the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ meets the criteria of the Alternating Series Test, it converges.
- Therefore, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges conditionally.

11-10. Alternating Series Error Bound

The **Alternating Series** Error Bound (also known as the Alternating Series Remainder) provides a way to estimate the error when approximating the sum of an alternating series by its partial sums. This error bound is useful for determining how close the partial sum is to the actual sum of the series.

Given an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$, where a_n are positive, decreasing, and $\lim_{n\to\infty} a_n = 0$, the error R_N when approximating the sum by the N $-thpartialsum$ S_N is bounded by the absolute value of the first omitted term:

$$|R_N| = \left|S - S_N\right| \le a_{N+1}$$

where S is the actual sum of the series and S_N is the N-th partial sum.

Formulas:

- Error Bound for Alternating Series: $\left|S - S_N\right| \le a_{N+1}$

where S is the actual sum, S_N is the N-th partial sum, and a_{N+1} is the $(N+1)$-th term of the series.

Examples:

1) Estimating the Sum of an Alternating Series

- Consider the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. Estimate the sum using the first 5 terms and provide an error bound.

- The given series is: $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$
- Calculate the 5th partial sum S_5: $S_5 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 1 - 0.5 + 0.333 - 0.25 + 0.2 = 0.783$
- The error bound is given by the absolute value of the 6th term: $|R_5| \le a_6 = \frac{1}{6} \approx 0.167$
- So, the actual sum S is within 0.167 of 0.783: $|S - 0.783| \le 0.167$

2) Estimating the Sum of Another Alternating Series

- Consider the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$. Estimate the sum using the first 4 terms and provide an error bound.

- The given series is: $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\cdots$
- Calculate the 4th partial sum S_4: $S_4=-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}=-1+0.25-0.111+0.0625=-0.7985$
- The error bound is given by the absolute value of the 5th term: $|R_4|\le a_5=\frac{1}{5^2}=\frac{1}{25}=0.04$
- So, the actual sum S is within 0.04 of -0.7985: $|S+0.7985|\le 0.04$

3) Estimating the Sum of a Converging Alternating Series

- Consider the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$. Estimate the sum using the first 3 terms and provide an error bound.

- The given series is: $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}=1-\frac{1}{2^3}+\frac{1}{3^3}-\cdots$
- Calculate the 3rd partial sum S_3: $S_3=1-\frac{1}{8}+\frac{1}{27}=1-0.125+0.037=0.912$
- The error bound is given by the absolute value of the 4th term: $|R_3|\le a_4=\frac{1}{4^3}=\frac{1}{64}\approx 0.0156$
- So, the actual sum S is within 0.0156 of 0.912: $|S-0.912|\le 0.0156$

11-11. Finding Taylor Polynomial Approximations of Functions

Taylor polynomials provide a way to approximate functions near a specific point using polynomials. These approximations are derived from the derivatives of the function at that point. The Taylor polynomial of degree n for a function $f(x)$ centered at $x = a$ is given by:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For $a = 0$, the Taylor polynomial is often referred to as the Maclaurin polynomial.

Formulas:

1. Taylor Polynomial of Degree n Centered at a :

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

where $f^{(k)}(a)$ denotes the k $-th derivative of $ f evaluated at a $.

2. Maclaurin Polynomial (Special case with $ a = 0):

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

Examples:

1) Taylor Polynomial of e^x Centered at $a=0$ (Maclaurin Series)

- Find the Maclaurin polynomial of degree 4 for $f(x)=e^x$.

- For $f(x)=e^x$, all derivatives are $f^{(k)}(x)=e^x$, and evaluated at $x=0$, we have $f^{(k)}(0)=1$.

- The Maclaurin polynomial of degree 4 is:

$$P_4(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

$$P_4(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$$

2) Taylor Polynomial of $\sin(x)$ Centered at $a=0$ (Maclaurin Series)

- Find the Maclaurin polynomial of degree 5 for $f(x)=\sin(x)$.

- For $f(x)=\sin(x)$, the derivatives cycle as follows:

$$f'(x)=\cos(x),\quad f''(x)=-\sin(x),\quad f'''(x)=-\cos(x),\quad f^{(4)}(x)=\sin(x),\quad \text{and so on}$$

- Evaluating at $x=0$:

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=-1,\quad f^{(4)}(0)=0,\quad f^{(5)}(0)=1$$

- The Maclaurin polynomial of degree 5 is: $P_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}$

$$P_5(x)=x-\frac{x^3}{6}+\frac{x^5}{120}$$

3) Taylor Polynomial of $\ln(1+x)$ Centered at $a=0$ (Maclaurin Series)

- Find the Maclaurin polynomial of degree 3 for $f(x)=\ln(1+x)$.

- For $f(x)=\ln(1+x)$, the derivatives are:

$$f'(x)=\frac{1}{1+x}, \quad f''(x)=-\frac{1}{(1+x)^2}, \quad f'''(x)=\frac{2}{(1+x)^3}$$

- Evaluating at $x=0$:

$$f(0)=\ln(1)=0, \quad f'(0)=1, \quad f''(0)=-1, \quad f'''(0)=2$$

- The Maclaurin polynomial of degree 3 is:

$$P_3(x)=0+x-\frac{x^2}{2!}+\frac{2x^3}{3!}$$

$$P_3(x)=x-\frac{x^2}{2}+\frac{x^3}{3}$$

4) Taylor Polynomial of $\cos(x)$ Centered at $a=\pi/4$

- Find the Taylor polynomial of degree 2 for $f(x)=\cos(x)$ centered at $a=\pi/4$.

- For $f(x)=\cos(x)$, the derivatives are:

$$f'(x)=-\sin(x), \quad f''(x)=-\cos(x), \quad f'''(x)=\sin(x)$$

- Evaluating at $x=\pi/4$:

$$f(\pi/4)=\cos(\pi/4)=\frac{\sqrt{2}}{2}, \quad f'(\pi/4)=-\sin(\pi/4)=-\frac{\sqrt{2}}{2}, \quad f''(\pi/4)=-\cos(\pi/4)=-\frac{\sqrt{2}}{2}$$

- The Taylor polynomial of degree 2 is:

$$P_2(x)=\cos(\pi/4)-\sin(\pi/4)(x-\pi/4)-\frac{\cos(\pi/4)}{2!}(x-\pi/4)^2$$

$$P_2(x)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}(x-\pi/4)-\frac{\sqrt{2}}{4}(x-\pi/4)^2$$

11-12. Lagrange Error Bound

The **Lagrange Error Bound** provides a way to estimate the error when using a Taylor polynomial to approximate a function. This error bound is particularly useful for understanding how closely the Taylor polynomial approximates the actual function.

For a function f that is approximated by its Taylor polynomial $P_n(x)$ of degree n centered at a , the Lagrange Error Bound gives an estimate for the error $R_n(x)$ at a point x within the interval of approximation. The error bound is given by:

$$|R_n(x)| = \left|f(x) - P_n(x)\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

where M is an upper bound for the absolute value of the $(n+1)$-th derivative of f on the interval containing a and x .

Examples:

1) Estimating the Error for e^x

- Approximate e^x using the second-degree Maclaurin polynomial $P_2(x) = 1 + x + \frac{x^2}{2}$ and find the error bound for $x = 0.1$.

- The Maclaurin polynomial of degree 2 for e^x is: $P_2(x) = 1 + x + \frac{x^2}{2}$
- The $(n+1)$-th derivative for e^x is also e^x . Since e^x is increasing, the maximum value of e^x on the interval [0, 0.1] occurs at $x = 0.1$: $M = e^{0.1} \approx 1.105$
- Apply the Lagrange Error Bound:

$$|R_2(0.1)| \le \frac{M}{3!}|0.1-0|^3 = \frac{1.105}{6}(0.1)^3 = \frac{1.105}{6000} \approx 0.000184$$

- So, the error bound for the approximation at $x = 0.1$ is approximately 0.000184.

2) Estimating the Error for $\sin(x)$

- Approximate $\sin(x)$ using the third-degree Maclaurin polynomial $P_3(x) = x - \frac{x^3}{6}$ and find the error bound for $x = 0.5$.

- The Maclaurin polynomial of degree 3 for $\sin(x)$ is: $P_3(x) = x - \frac{x^3}{6}$
- The $(n+1)$-th derivative for $\sin(x)$ is $\cos(x)$ or $-\cos(x)$. The maximum value of $|\cos(x)|$ on the interval $[-0.5, 0.5]$ is 1.
- Apply the Lagrange Error Bound:

$$|R_3(0.5)| \le \frac{M}{4!}|0.5-0|^4 = \frac{1}{24}(0.5)^4 = \frac{1}{24}\cdot\frac{1}{16} = \frac{1}{384} \approx 0.0026$$

- So, the error bound for the approximation at $x = 0.5$ is approximately 0.0026.

3) Estimating the Error for $\ln(1+x)$

Approximate $\ln(1+x)$ using the second-degree Taylor polynomial centered at $a = 0$, $P_2(x) = x - \frac{x^2}{2}$, and find the error bound for $x = 0.2$.

- The Taylor polynomial of degree 2 for $\ln(1+x)$ is: $P_2(x) = x - \frac{x^2}{2}$
- The $(n+1)$-th derivative for $\ln(1+x)$ is $\frac{(-1)^n}{(1+x)^n}$.
- The maximum value of $\left|\frac{(-1)^3}{(1+x)^3}\right| = \frac{1}{(1+x)^3}$ on the interval [0, 0.2] is at $x = 0$: $M = \frac{1}{1^3} = 1$
- Apply the Lagrange Error Bound:

$$|R_2(0.2)| \le \frac{M}{3!}|0.2-0|^3 = \frac{1}{6}(0.2)^3 = \frac{1}{6}\cdot 0.008 = \frac{0.008}{6} \approx 0.00133$$

- So, the error bound for the approximation at $x = 0.2$ is approximately 0.00133.

11-13. Understanding Euler's Method

Euler's Method is a numerical technique used to approximate solutions to ordinary differential equations (ODEs). Given an initial value problem of the form:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Euler's Method approximates the solution by iterating over small steps, using the slope at each point to estimate the next point.

Steps:

1. Starting Point: Begin with the initial condition (x_0, y_0).
2. Step Size: Choose a step size h.
3. Iteration: Use the following formula to find successive points:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$$x_{n+1} = x_n + h$$

This process is repeated for the desired number of steps or until a specific value of x is reached.

Summary:

Euler's Method is a straightforward numerical approach for approximating solutions to ordinary differential equations. By iterating with a small step size, it provides a sequence of approximations that converge to the actual solution as the step size decreases.

Examples:

1) Approximating y for a Simple Differential Equation

- Use Euler's Method to approximate the solution to $\frac{dy}{dx} = x + y$ with $y(0) = 1$ at $x = 0.2$ using a step size of $h = 0.1$.

- Initial condition: $x_0 = 0$, $y_0 = 1$
- Step size: $h = 0.1$

- Iteration 1:
  - $f(x_0, y_0) = f(0,1) = 0 + 1 = 1$
  - $y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0.1 \cdot 1 = 1.1$
  - $x_1 = x_0 + h = 0 + 0.1 = 0.1$
- Iteration 2:
  - $f(x_1, y_1) = f(0.1, 1.1) = 0.1 + 1.1 = 1.2$
  - $y_2 = y_1 + h \cdot f(x_1, y_1) = 1.1 + 0.1 \cdot 1.2 = 1.22$
  - $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
- Therefore, the approximate value of y at $x = 0.2$ is $y \approx 1.22$.

2) Approximating y for a Differential Equation with Nonlinear Terms

- Use Euler's Method to approximate the solution to $\frac{dy}{dx} = y - x^2 + 1$ with $y(0) = 0.5$ at $x = 0.2$ using a step size of $h = 0.1$.

- Initial condition: $x_0 = 0$, $y_0 = 0.5$
- Step size: $h = 0.1$
- Iteration 1:
  - $f(x_0, y_0) = f(0, 0.5) = 0.5 - 0^2 + 1 = 1.5$
  - $y_1 = y_0 + h \cdot f(x_0, y_0) = 0.5 + 0.1 \cdot 1.5 = 0.65$
  - $x_1 = x_0 + h = 0 + 0.1 = 0.1$
- Iteration 2:
  - $f(x_1, y_1) = f(0.1, 0.65) = 0.65 - (0.1)^2 + 1 = 1.65 - 0.01 = 1.64$
  - $y_2 = y_1 + h \cdot f(x_1, y_1) = 0.65 + 0.1 \cdot 1.64 = 0.814$
  - $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
- Therefore, the approximate value of y at $x = 0.2$ is $y \approx 0.814$.

11-14. Finding Taylor or Maclaurin Series for a Function

The **Taylor series** of a function provides a polynomial approximation of the function around a point a. When the point a is zero, the series is known as the Maclaurin series. These series are powerful tools for approximating functions using an infinite sum of terms derived from the function's derivatives at a single point.

Formulas:

1. Taylor Series: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ where $f^{(n)}(a)$ is the n-th derivative of f evaluated at $x=a$

2. Maclaurin Series (special case of the Taylor series at $a=0$): $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

Examples:

1) Maclaurin Series for e^x

- Find the Maclaurin series for $f(x)=e^x$.

- The function e^x and all its derivatives are e^x. At $x=0$, we have $f^{(n)}(0)=1$ for all n.
- The Maclaurin series is: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

2) Taylor Series for $\sin(x)$ **Centered at** $a = \pi / 4$

- Find the Taylor series for $f(x) = \sin(x)$ centered at $a = \pi / 4$.

- The derivatives of $\sin(x)$ cycle as follows:

$$f(x) = \sin(x),\ f'(x) = \cos(x),\ f''(x) = -\sin(x),\ f'''(x) = -\cos(x)$$

- Evaluating at $x = \pi / 4$:

$$f(\pi / 4) = \sin(\pi / 4) = \frac{\sqrt{2}}{2},\ f'(\pi / 4) = \cos(\pi / 4) = \frac{\sqrt{2}}{2}$$
$$f''(\pi / 4) = -\sin(\pi / 4) = -\frac{\sqrt{2}}{2},\ f'''(\pi / 4) = -\cos(\pi / 4) = -\frac{\sqrt{2}}{2}$$

- The Taylor series is:

$$\sin(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \pi / 4) - \frac{\sqrt{2}}{4!}(x - \pi / 4)^2 - \frac{\sqrt{2}}{3!}(x - \pi / 4)^3 + \cdots$$

3) Maclaurin Series for $\cos(x)$

- Find the Maclaurin series for $f(x) = \cos(x)$.

- The derivatives of $\cos(x)$ cycle as follows:

$$f(x) = \cos(x), \quad f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f'''(x) = \sin(x)$$

- Evaluating at $x = 0$:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0$$

- The Maclaurin series is:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

11-15. Representing Functions as Power Series

A **power series** is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where c_n are the coefficients, a is the center of the series, and x is the variable. Power series can represent functions within their radius of convergence. Many functions can be expressed as power series, allowing for powerful techniques in analysis and approximation.

Theorems and Tests:

1. **Radius of Convergence**: The radius of convergence R of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is found using:

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$$

or the Ratio Test: $R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$

2. **Interval of Convergence:** The interval of convergence is the set of x $-values for which the series converges. This interval is (a-R, a+R) , and endpoints must be checked separately.

Examples:

1) Power Series Representation of $\frac{1}{1-x}$

- Find the power series representation of $f(x)=\frac{1}{1-x}$ centered at $a=0$.

- The function $\frac{1}{1-x}$ can be expressed as a geometric series for $|x|<1$:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

- The radius of convergence R is 1, and the interval of convergence is $(-1,1)$.

2) Power Series Representation of e^x

- Find the power series representation of $f(x)=e^x$ centered at $a=0$.

- The function e^x and all its derivatives are e^x. The Maclaurin series is:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

- The radius of convergence R is infinite, and the interval of convergence is $(-\infty,\infty)$.

3) Power Series Representation of $\sin(x)$

- Find the power series representation of $f(x)=\sin(x)$ centered at $a=0$.

- The Maclaurin series for $\sin(x)$ is:

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

- The radius of convergence R is infinite, and the interval of convergence is $(-\infty,\infty)$.