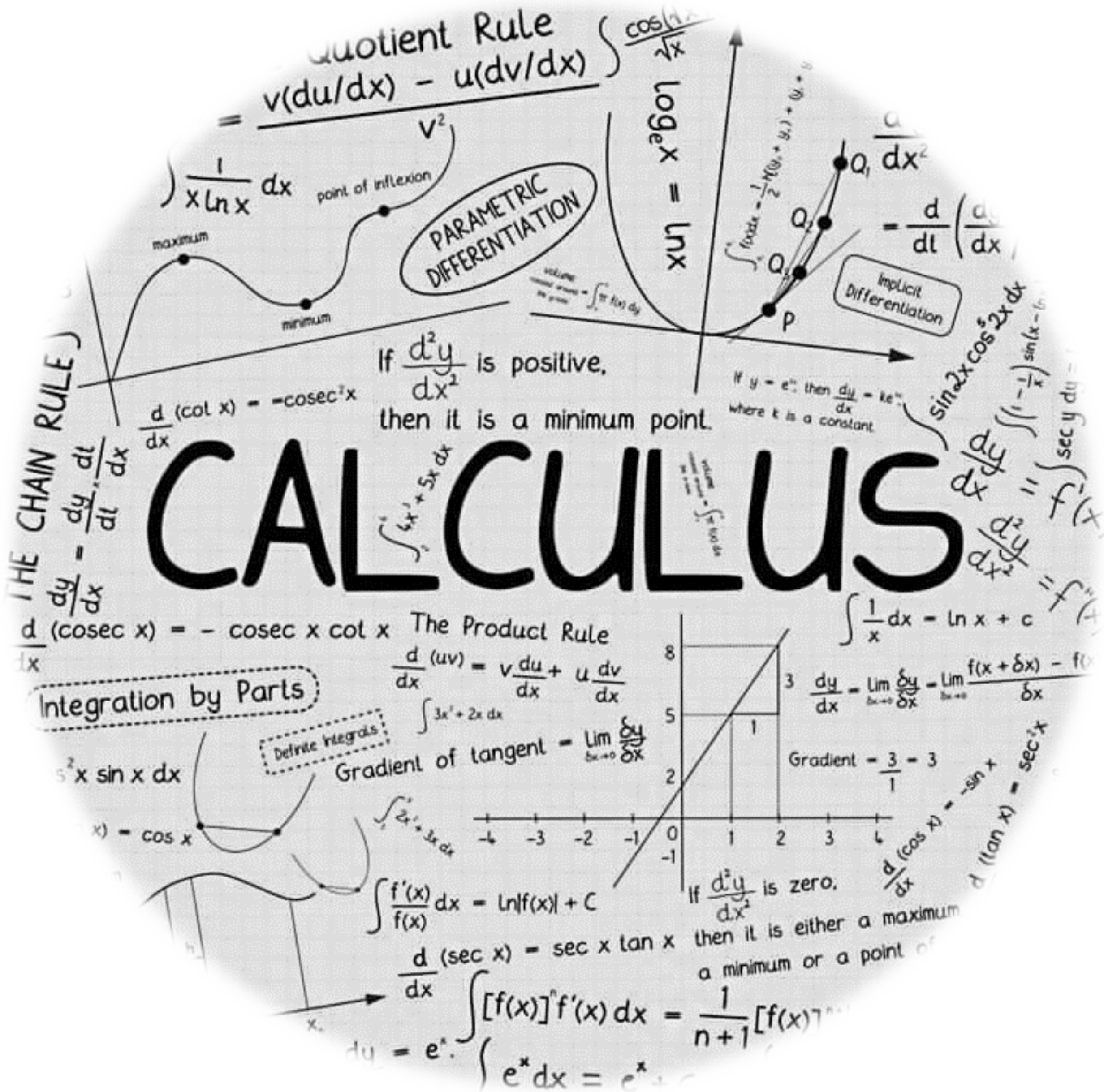
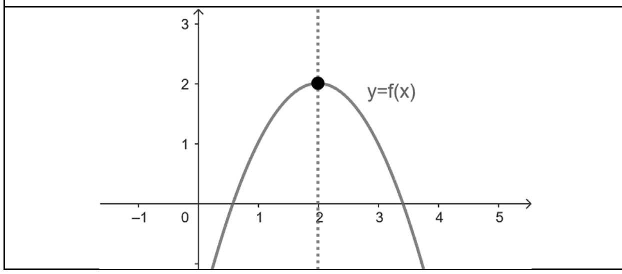
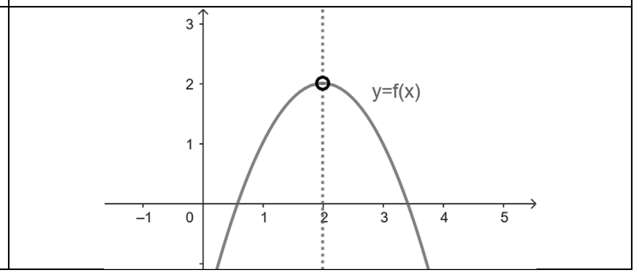
Contents

Chapter 0. Foundations and Previews

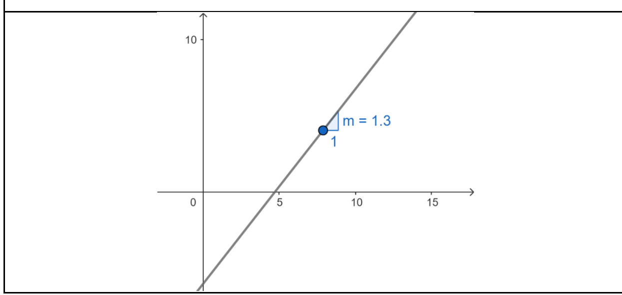
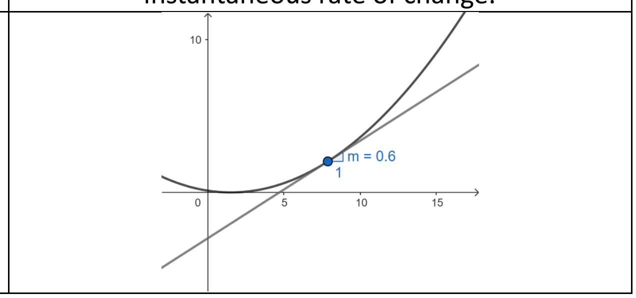


0-1. Why Calculus

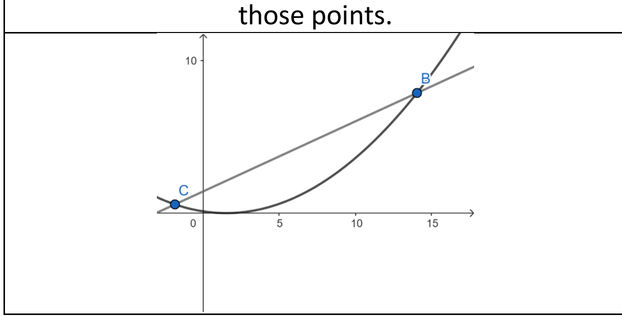
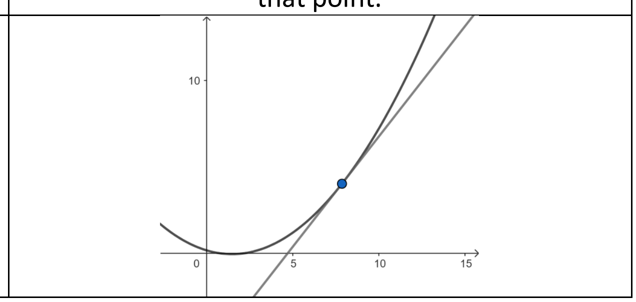
1. Value of $f(x)$ when $x = c$:

<table>
<tr><td>Without Calculus</td><td>With Differential Calculus</td></tr>
<tr><td>You directly find the value of the function f at $x = c$.</td><td>You consider the limit of $f(x)$ as x approaches c, which can be more precise, especially if f is not continuous at c.</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

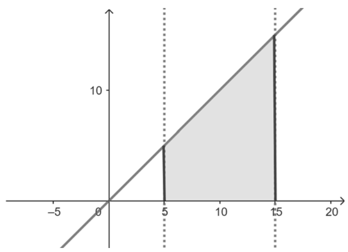
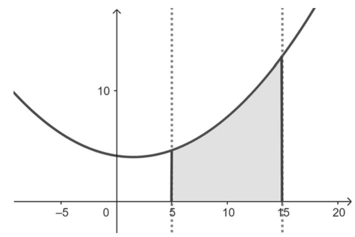
2. Slope of a line:

<table>
<tr><td>Without Calculus</td><td>With Differential Calculus</td></tr>
<tr><td>The slope is the change in y divided by the change in x ($\Delta y / \Delta x$).</td><td>The slope of a curve at a point is found using the derivative (dy / dx), representing the instantaneous rate of change.</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

3. Secant line to a curve:

<table>
<tr><td>Without Calculus</td><td>With Differential Calculus</td></tr>
<tr><td>A secant line intersects the curve at two points, representing the average rate of change between those points.</td><td>A tangent line touches the curve at one point, representing the instantaneous rate of change at that point.</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

4. Area under the line or curve:

Without Calculus	With Differential Calculus
You find the area by multiplying the length and width of the rectangle/polygon.	**You find the area under a curve** using integration, which can handle more complex shapes.
	

5. Length of a line segment:

Without Calculus	With Differential Calculus
The length is the distance between two points.	You find the **length of an arc** (curved line) using integration.

6. Surface area of a cylinder:

Without Calculus	With Differential Calculus
You calculate the surface area using the formula for a cylinder.	You find **the surface area of a solid** of revolution using integration, which can handle more complex shapes.

7. Mass of a solid of constant density:

Without Calculus	With Differential Calculus
The mass is found by multiplying the volume by the constant density.	You calculate the **mass of a solid** with variable density using integration.

8. Volume of a rectangular solid:

Without Calculus	With Differential Calculus
The volume is found by multiplying length, width, and height.	You find **the volume of a region under a surface** using integration.

0-2. Increasing & Decreasing functions (use in curve sketching)

1) Determine the domain and range of the function $g(x)=2e^{x}$.	2) Investigate the properties of the function $h(x)=e^{x-2}$. Specifically, determine the domain, range, and whether the function is increasing or decreasing.
3) Determine the domain and range of the function $g(x)=2\sin(x)$.	4) Investigate the properties of the function $h(x)=\sin(x-\frac{\pi}{4})$. Specifically, determine the domain, range, and symmetry.
5) Determine the domain and range of the function $g(x)=\frac{1}{2x}$.	6) Investigate the properties of the function $h(x)=\frac{1}{x-3}$. Specifically, determine the domain, range, and symmetry.

Solutions:

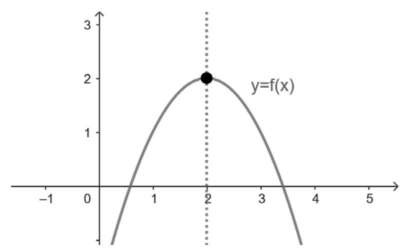
1) Determine the domain and range of the function $g(x)=2e^{x}$ **.** The function $g(x)=2e^{x}$ is a transformation of the basic exponential function e^{x} . - Domain: The domain of e^{x} is all real numbers, $(-\infty,\infty)$. Since $g(x)=2e^{x}$ is just a vertical stretch, the domain remains the same. - Domain: $(-\infty,\infty)$ - Range: The range of . e^{x} . is $(0,\infty)$. Multiplying by 2 stretches the range but does not change its lower or upper bounds. - Range: $(0,\infty)$	**2) Investigate the properties of the function** $h(x)=e^{x-2}$ **. Specifically, determine the domain, range, and whether the function is increasing or decreasing.** The function $h(x)=e^{x-2}$ is a horizontal shift of the basic exponential function e^{x} . - Domain: The domain of e^{x} is all real numbers, $(-\infty,\infty)$. The horizontal shift does not affect the domain. - Domain: $(-\infty,\infty)$ - Range: The range of e^{x} is $(0,\infty)$. The horizontal shift does not affect the range. - Range: $(0,\infty)$ - Increasing/Decreasing: The function e^{x} is always increasing, and the horizontal shift does not change this property. - Increasing on $(-\infty,\infty)$
3) Determine the domain and range of the function $g(x)=2\sin(x)$ **.** The function $g(x)=2\sin(x)$ is a vertical stretch of the basic sine function $\sin(x)$. - Domain: The domain of $\sin(x)$ is all real numbers, $(-\infty,\infty)$. The vertical stretch does not affect the domain. - Domain: $(-\infty,\infty)$ - Range: The range of $\sin(x)$ is $[-1,1]$. Multiplying by 2 stretches the range to $[-2,2]$. - Range: $[-2,2]$	**4) Investigate the properties of the function** $h(x)=\sin(x-\frac{\pi}{4})$ **. Specifically, determine the domain, range, and symmetry.** The function $h(x)=\sin(x-\frac{\pi}{4})$ is a horizontal shift of the basic sine function $\sin(x)$. - Domain: The domain of $\sin(x)$ is all real numbers, $(-\infty,\infty)$. The horizontal shift does not affect the domain. - Domain: $(-\infty,\infty)$ - Range: The range of $\sin(x)$ is $[-1,1]$. The horizontal shift does not affect the range. - Range: $[-1,1]$ - Symmetry: The function $\sin(x)$ is an odd function, symmetric about the origin. However,

	$h(x)=\sin(x-\frac{\pi}{4})$ is not an odd/even function since it is a horizontally shifted sine function. - The sine function has a period of 2π, so the function $h(x)=\sin\left(x-\frac{\pi}{4}\right)$ also has a period of 2π.
5) Determine the domain and range of the function $g(x)=\frac{1}{2x}$. The function $g(x)=\frac{1}{2x}$ is a vertical compression of the basic function $\frac{1}{x}$. - - Domain: The domain of $\frac{1}{x}$ is all real numbers except $x=0$, because division by zero is undefined. The vertical compression does not affect the domain. - Domain: $(-\infty,0)\cup(0,\infty)$ - - Range: The range of $\frac{1}{x}$ is all real numbers except $y=0$, since $\frac{1}{x}$ never equals zero. The vertical compression does not affect the range. - Range: $(-\infty,0)\cup(0,\infty)$	**6) Investigate the properties of the function $h(x)=\frac{1}{x-3}$. Specifically, determine the domain, range, and symmetry.** The function $h(x)=\frac{1}{x-3}$ is a horizontal shift of the basic function $\frac{1}{x}$. - Domain: The domain of $\frac{1}{x}$ is all real numbers except $x=0$. For $\frac{1}{x-3}$, the horizontal shift means the function is undefined at $x=3$. - Domain: $(-\infty,3)\cup(3,\infty)$ - Range: The range of $\frac{1}{x}$ is all real numbers except $y=0$. The horizontal shift does not affect the range. - Range: $(-\infty,0)\cup(0,\infty)$ - Symmetry: The function $\frac{1}{x}$ is an odd function, symmetric about the origin. However, the horizontal shift breaks this symmetry. - Symmetry: None

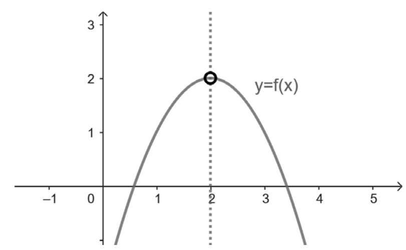
0-3. Limit Foundation

Find the limit?

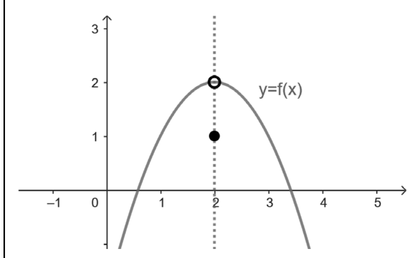
1) $\lim_{x \to 2} f(x)$



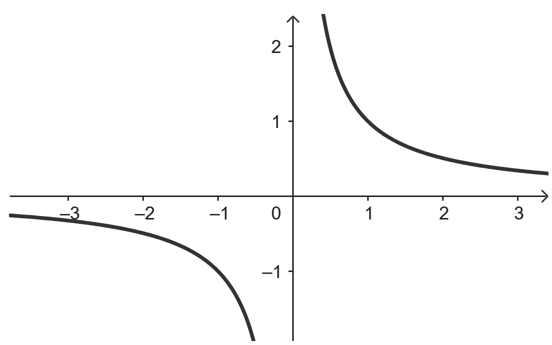
2) $\lim_{x \to 2} f(x)$



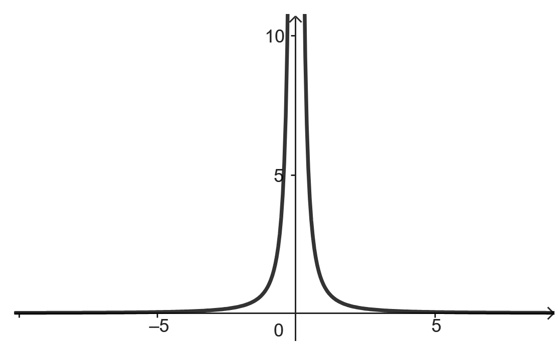
3) $\lim_{x \to 2} f(x)$



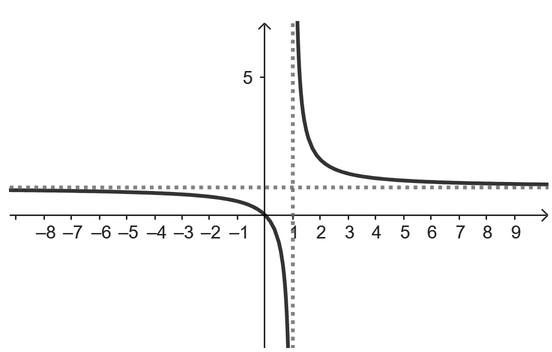
4) $\lim_{x \to 0^-} \frac{1}{x} =$ 5) $\lim_{x \to 0^+} \frac{1}{x} =$



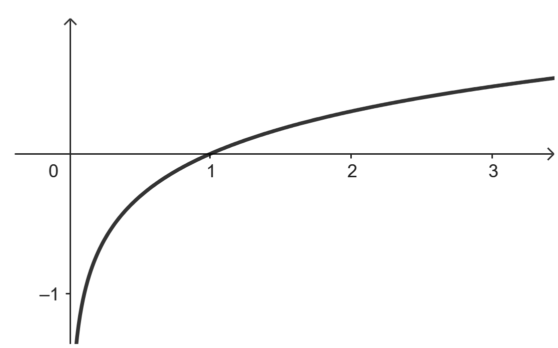
6) $\lim_{x \to 0^-} \frac{1}{x^2} =$ 7) $\lim_{x \to 0^+} \frac{1}{x^2} =$



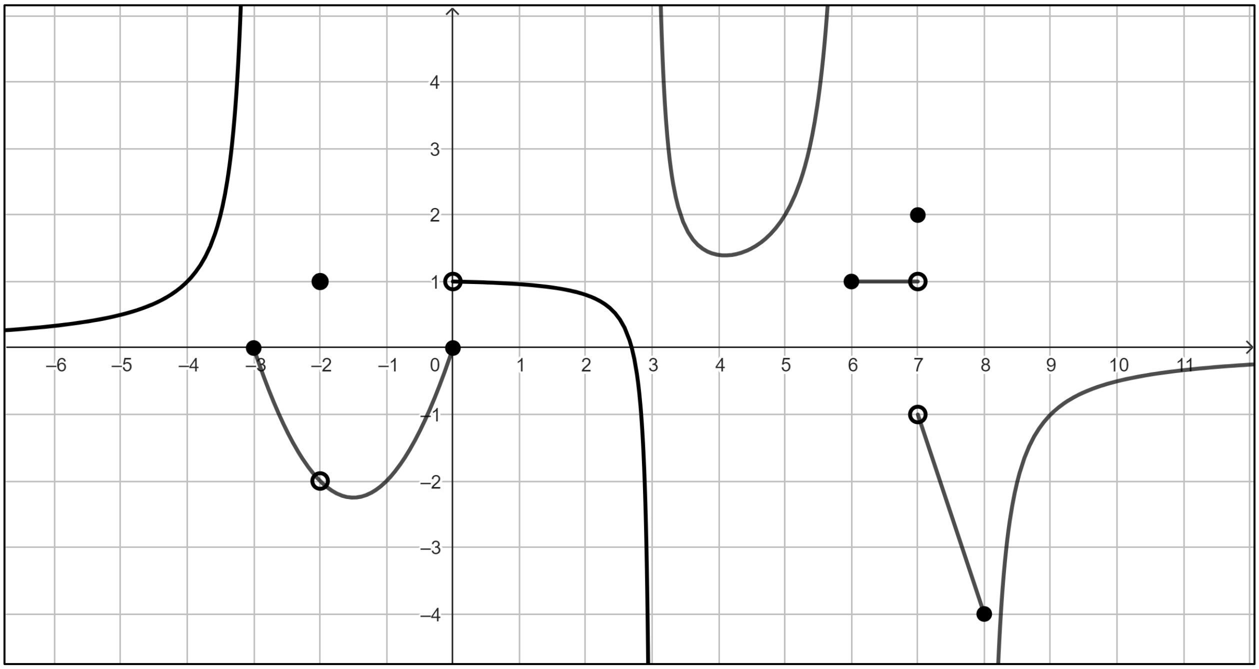
8) $\lim_{x \to 1^-} \frac{x}{x-1} =$ 9) $\lim_{x \to 1^+} \frac{x}{x-1} =$



10) $\lim_{x \to \infty} \log(x) =$ 11) $\lim_{x \to 0^+} \log(x) =$



12) Given the graph of $f(x)$ above, find the limit.



$\lim_{x\to-\infty} f(x)=$	$\lim_{x\to\infty} f(x)=$	$\lim_{x\to-3^+} f(x)=$	$\lim_{x\to-3^-} f(x)=$
$\lim_{x\to-2^-} f(x)=$	$\lim_{x\to-2^+} f(x)=$	$\lim_{x\to0^+} f(x)=$	$\lim_{x\to0^-} f(x)=$
$\lim_{x\to3^+} f(x)=$	$\lim_{x\to3^-} f(x)=$	$\lim_{x\to6^+} f(x)=$	$\lim_{x\to6^-} f(x)=$
$\lim_{x\to7^+} f(x)=$	$\lim_{x\to7^-} f(x)=$	$\lim_{x\to8^+} f(x)=$	$\lim_{x\to8^-} f(x)=$
$f(-2)=$	$f(7)=$		

Solutions:

1) $\lim_{x\to 2} f(x)=2$	2) $\lim_{x\to 2} f(x)=2$	3) $\lim_{x\to 2} f(x)=2$

4) $\lim_{x\to 0^-} \frac{1}{x}=-\infty$	5) $\lim_{x\to 0^+} \frac{1}{x}=\infty$	6) $\lim_{x\to 0^-} \frac{1}{x^2}=\infty$	7) $\lim_{x\to 0^+} \frac{1}{x^2}=\infty$

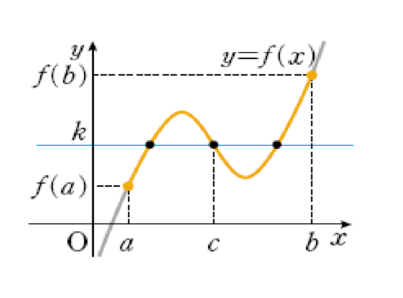
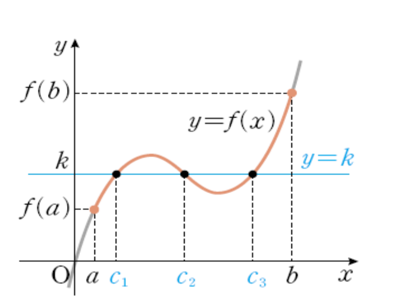
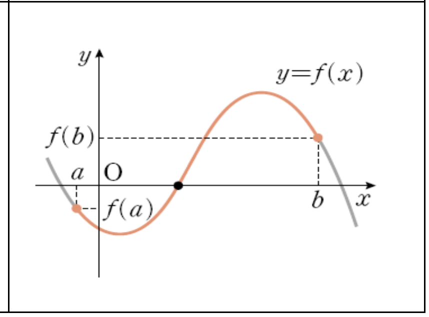
8) $\lim_{x\to 1^-} \frac{x}{x-1}=-\infty$	9) $\lim_{x\to 1^+} \frac{x}{x-1}=\infty$	10) $\lim_{x\to \infty} \log(x)=0$	11) $\lim_{x\to 0^+} \log(x)=-\infty$

12)			
$\lim_{x\to -\infty} f(x)=0$	$\lim_{x\to \infty} f(x)=0$	$\lim_{x\to -3^+} f(x)=0$	$\lim_{x\to -3^-} f(x)=\infty$
$\lim_{x\to -2^-} f(x)=-2$	$\lim_{x\to -2^+} f(x)=-2$	$\lim_{x\to 0^+} f(x)=1$	$\lim_{x\to 0^-} f(x)=0$
$\lim_{x\to 3^+} f(x)=\infty$	$\lim_{x\to 3^-} f(x)=-\infty$	$\lim_{x\to 6^+} f(x)=1$	$\lim_{x\to 6^-} f(x)=-\infty$
$\lim_{x\to 7^+} f(x)=-1$	$\lim_{x\to 7^-} f(x)=1$	$\lim_{x\to 8^+} f(x)=-\infty$	$\lim_{x\to 8^-} f(x)=-4$
$f(-2)=1$	$f(7)=2$		

0-4. Basic Theorems

Theorem: Intermediate Value Theorem, IVT

If **f is continuous** on a closed interval [a, b] and k is any number between f(a) and f(b), then there exists at least one number c in (a, b) such that f(c) = k.

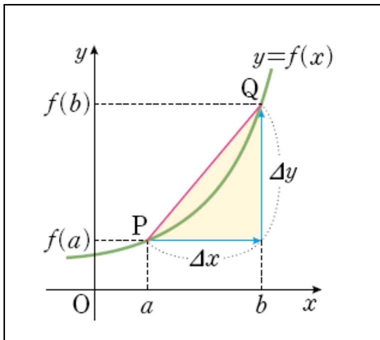
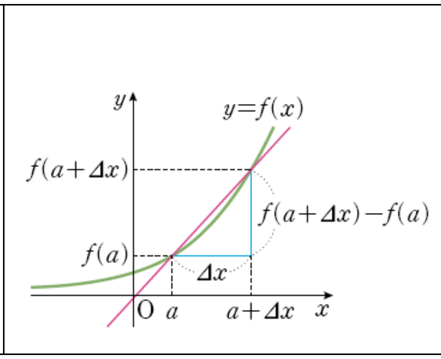
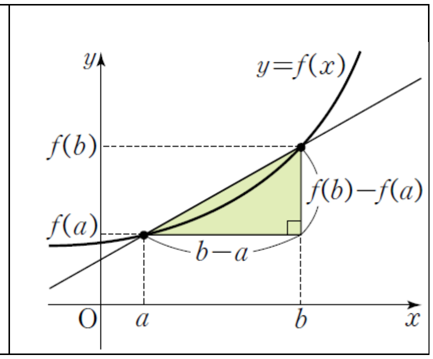
		If f(b)>0 and f(a)<0, then there exists at least one solution.
		

Definition of the Average Rate of Change

The average rate of change of y (slope m) with respect to x over the interval $[a,b]$ is given by:

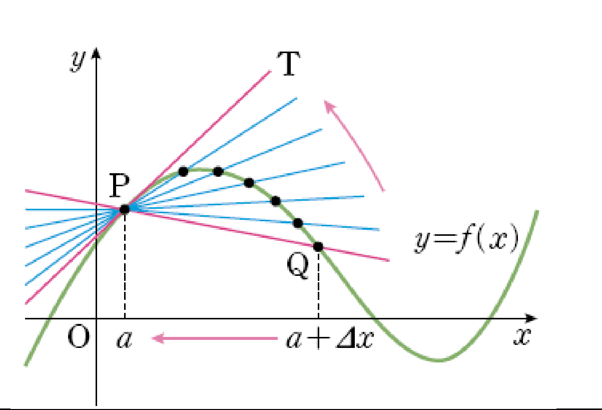
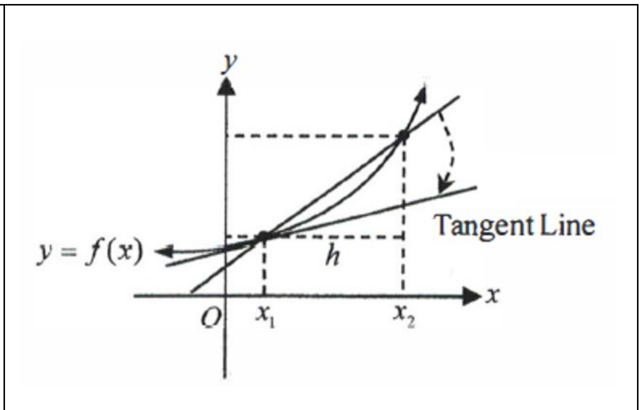
$$m=\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}=\frac{f(a+\Delta x)-f(a)}{\Delta x}=\frac{f(a+h)-f(a)}{h}$$

where $h=\Delta x=b-a$.

Definition of the Instant Rate of Change

The graph demonstrates the concept of secant lines approaching the tangent line at a specific point on a curve as the interval between the points on the secant line (Δx) approaches zero. This visual representation helps in understanding the definition of the derivative, which is the slope of the tangent line at a given point on the function.

	
$f'(a)=\lim_{\Delta x\to 0}\dfrac{f(a+\Delta x)-f(a)}{\Delta x}$	$f'(x_1)=\lim_{x_2\to x_1}\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

- **Secant Lines**: The lines passing through points P and Q are secant lines. These lines intersect the curve at two points and approximate the slope of the function between those points.

  - As Δx decreases (meaning Q moves closer to P), the secant lines approach the slope of the tangent line at P.

- **Tangent Line T**:

  - Definition: The tangent line at point P is the line that just touches the curve at P without crossing it. This line represents the instantaneous rate of change of the function at $x=a$.

  - Slope of the Tangent Line: The slope of the tangent line at P is the limit of the slopes of the secant lines as Δx approaches zero.

- The slope of the tangent line at $x=a$ is given by the derivative: (replace Δx to h)

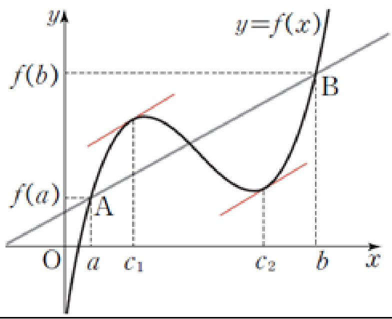
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

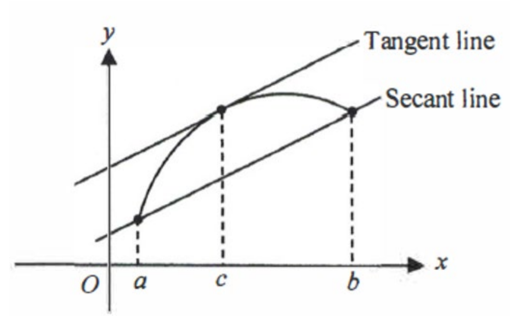
THEOREM: The Mean-Value Theorem (MVT)

If f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number c between a and b such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

The slope of the secant line is equal to the slope of the tangent line.





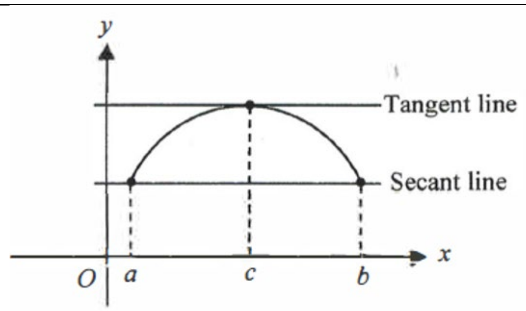
The Mean-Value Theorem guarantees that there is at least one point c in the interval (a, b) where the tangent line has the same slope as the secant line.

THEOREM: Rolle's Theorem (Special case of the Mean Value Theorem)

Let f be continuous on the closed interval [a, b] and differentiable on the open interval (a, b).

If $f(b)=f(a)$, then there exists at least one number c between a and b such that

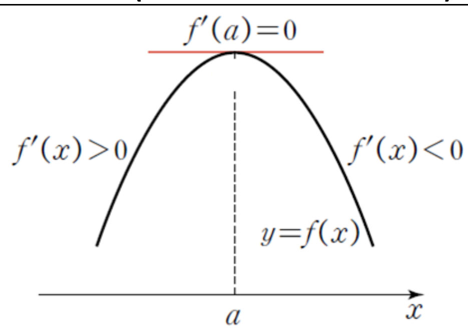
$$\frac{f(b)-f(a)}{b-a}=f'(c)=0.$$

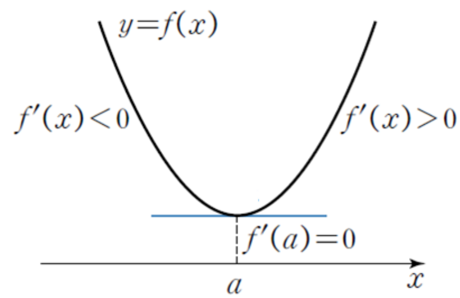


Rolle's Theorem will guarantee the existence of an extreme value (relative maximum or relative minimum) in the interval.

Maxima and Minima

- Maxima: At a local maximum, the derivative changes from positive to negative.
- Minima: At a local minimum, the derivative changes from negative to positive.
- Tangent lines at the critical points where $f'(a)=0$ confirm the behavior of the slopes, showing a peak for maxima and a valley for minima.

<table>
<tr><th>Maxima (relative maximum at x=a)</th><th>Minima (relative minimum at x=a)</th></tr>
<tr><td>


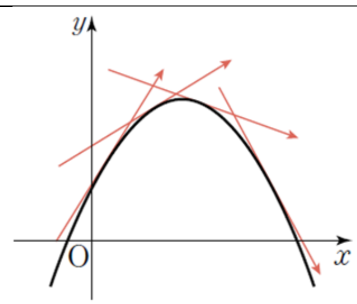
</td><td>


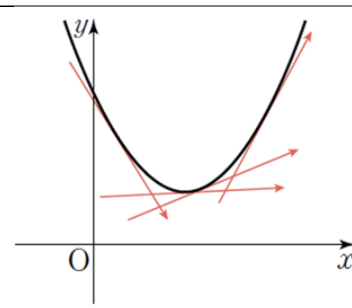
</td></tr>
<tr><td>

- Function $y=f(x)$: The curve represents the function.
- Critical Point at a: The point where the slope of the tangent is zero, $f'(a)=0$.
- Left of a: $f'(x)>0$, the function is increasing.
- Right of a: $f'(x)<0$, the function is decreasing.
- This indicates a local maximum at $x=a$.

</td><td>

- Function $y=f(x)$: The curve represents the function.
- Critical Point at a: The point where the slope of the tangent is zero, $f'(a)=0$.
- Left of a: $f'(x)<0$, the function is decreasing.
- Right of a: $f'(x)>0$, the function is increasing.
- This indicates a local minimum at $x=a$.

</td></tr>
<tr><td>


</td><td>


</td></tr>
<tr><td>

- Shows the tangent lines with positive slopes approaching $x=a$ from the left and negative slopes after $x=a$, confirming the maximum.
- Concave Downward

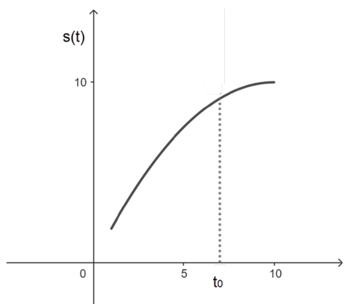
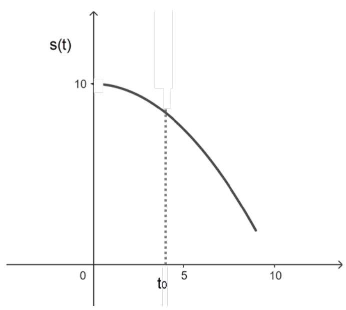
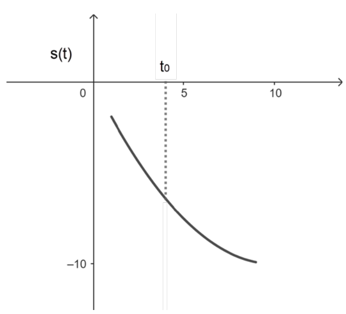
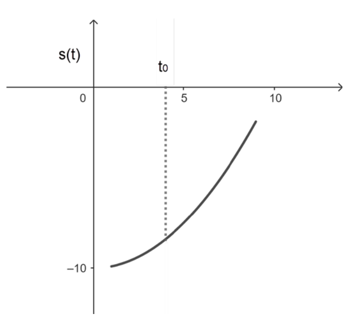
</td><td>

- Shows the tangent lines with negative slopes approaching $x=a$ from the left and positive slopes after $x=a$, confirming the minimum.
- Concave Upward

</td></tr>
</table>

0-5. Behavior of the Particle about Position vs. Time Curve

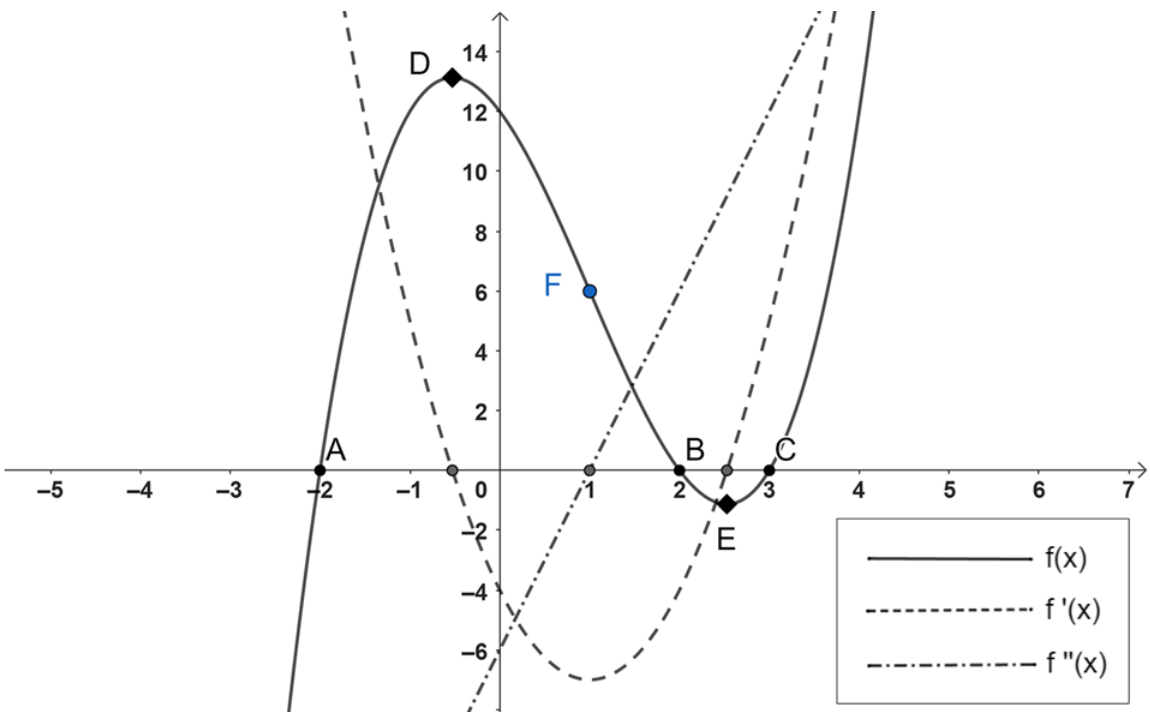
Observe behavior of the particle about the position versus time curve.

	At $t = t_0$ - Curve has positive slope. - Curve is concave down. - $s(t_0) > 0$ - $s'(t_0) = v(t_0) > 0$ - $s''(t_0) = a(t_0) < 0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is decreasing. - Particle is slowing down. - $v(t_0) > 0$ and $a(t_0) < 0$
	2. At $t = t_0$ - Curve has negative slope. - Curve is concave down. - $s(t_0) > 0$ - $s'(t_0) = v(t_0) < 0$ - $s''(t_0) = a(t_0) < 0$	- Particle is on the positive side of the origin. - Particle is moving in the negative direction. - Velocity is decreasing. - Particle is speeding up. - $v(t_0) < 0$ and $a(t_0) < 0$
	3. At $t = t_0$ - Curve has negative slope. - Curve is concave up. - - $s'(t_0) = v(t_0) < 0$ - $s''(t_0) = a(t_0) > 0$	- Particle is on the negative side of the origin. - Particle is moving in the negative direction. - Velocity is increasing. $s(t_0) < 0$ - Particle is slowing down. - $v(t_0) < 0$ and . $a(t_0) > 0$.
	4. At $t = t_0$ - Curve has positive slope. - Curve is concave up. - $s(t_0) < 0$ - $s'(t_0) = v(t_0) > 0$ - $s''(t_0) = a(t_0) > 0$	- Particle is on the positive side of the origin. - Particle is moving in the positive direction. - Velocity is increasing. - Particle is speeding up. - $v(t_0) > 0$ and $a(t_0) > 0$

0-6. Derivative Test

Concept Expansion from Pre-Calculus:

Can you find local maximum (D) and local minimum (E) values and the inflection point (F) for $f(x) = x^3 - 3x^2 - 4x + 12$ without using Calculus Concept?



Solution)

- We can easily find roots (A, B & C) by factorization $\Rightarrow roots: x = -2, 2, 3$
- However, it is not easy to find x values for D (local max), E(local min) and F(Inflection point). Before calculus, to solve this problem, we may need to use the approximation method.
- Once we learn about derivatives, then we can find these points easily.

Practice Example: Sketch the polynomial graph of $f(x)=x^3-3x^2-24x+32$ by using $f'(x)$, $f''(x)$

Solution Steps:

1. Find $f(x)$, $f'(x)$, $f''(x)$

- $f(x)=x^3-3x^2-24x+32$
- $f'(x)=3x^2-6x-24$
- $f''(x)=6x-6$

2. Find the first derivative ($f'(x)$) equal to zero to find **critical points** and its functional values if exist

- $3x^2-6x-24=0 \Rightarrow 3(x-4)(x+2)=0 \Rightarrow x=4, x=-2$
- $f(-2)=(-2)^3-3(-2)^2-24(-2)+32=60$ (maxima)
- $f(4)=(4)^3-3(4)^2-24(4)+32=-48$ **(minima)**

3. Set the second derivative **(** $f''(x)$ **)** equal to zero to find **inflection points** and its functional values if exist

- $6x-6=0 \Rightarrow 6(x-1)=0 \Rightarrow x=1$
- $f(1)=(1)^3-3(1)^2-24(1)+32=6$ **(inflection point)**

4. Determine the y-intercept

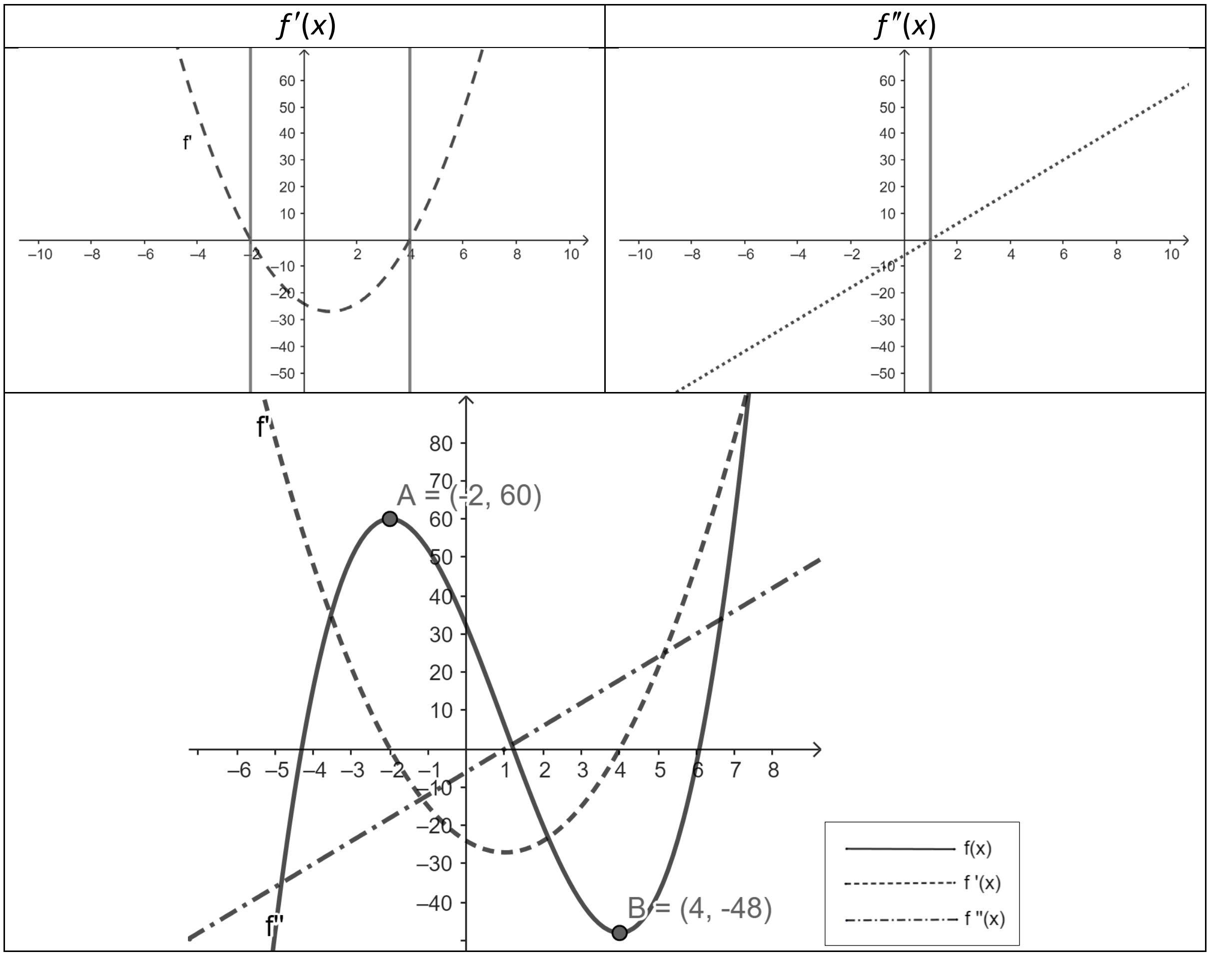
- $f(0)=0^3-3(0)^2-24(0)+32=32$

5. Determine the **concavity and relative extrema** using the first and second derivatives

Number of critical points (including inflection points): 3, so need 4 sections on the graph

$f(x)=x^3-3x^2-24x+32$							
$f'(x)=3x^2-6x-24$	+	-2	-		-	4	+
$f''(x)=6x-6$	-		-	1	+		+

6. Sketch the graph:



1. **Plot the relative maximum at A** $(-2,60)$**.**
2. **Plot the relative minimum at B** $(4,-48)$**.**
3. **Plot the inflection point at C** $(1,6)$**.**
4. **Plot the y-intercept at D** $(0,32)$**.**
5. **Draw the curve concave down from** $(-\infty,-2)$**, then continue concave down through** $(-2,60)$ **to** $(1,6)$**.**
6. **Switch to concave up from** $(1,6)$ **to** $(4,-48)$ **and continue concave up to** (∞,∞)**.**

Practice Example: Sketch the rational function graph of $f(x)=\dfrac{x^2-4x+3}{x}$ by using $f'(x)$, $f''(x)$

To sketch the rational function $f(x)=\dfrac{x^2-4x+3}{x}$ using its first and second derivatives, follow these steps:

1. Simplify the Function: $f(x)=\dfrac{x^2-4x+3}{x}=x-4+\dfrac{3}{x}$

2. Find Asymptotes

- Vertical Asymptote: Occurs where the denominator is zero: $x=0$
- Slanted (oblique) Asymptote: $y=x-4$

3. Find Intercepts

- x-intercepts: Set $f(x)=0$: $x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0$, So, $x=1$ and $x=3$.
- y-intercept: Set $x=0$: The function is undefined at $x=0$, so there is no y-intercept.

4. Find Critical Points (First Derivative)

- Find the first derivative $f'(x)$: $f'(x)=\dfrac{d}{dx}\left(x-4+\dfrac{3}{x}\right)=1-\dfrac{3}{x^2}$
- Set $f'(x)=0$: $1-\dfrac{3}{x^2}=0$ $x=\pm\sqrt{3}$

5. Find Points of Inflection (Second Derivative)

- Find the second derivative $f''(x)$: $f''(x)=\dfrac{d}{dx}\left(1-\dfrac{3}{x^2}\right)=\dfrac{6}{x^3}$
- Set $f''(x)=0$: $\dfrac{6}{x^3}=0$
- There are no real solutions. So, there are no points of inflection.

6. Analyze Intervals of Increase/Decrease

- For $x>0$:

- $f'(x)=1-\dfrac{3}{x^2}$

- If $x > \sqrt{3}$, $f'(x) > 0$ (positive).

- If $0 < x < \sqrt{3}$, $f'(x) < 0$ (negative).

- For $x < 0$:

- $f'(x) = 1 - \frac{3}{x^2}$

- Always $f'(x) < 0$ (negative).

So, $x = -\sqrt{3}$ and $x = \sqrt{3}$ are **critical points:**

- Increasing on $(\sqrt{3}, \infty)$

- Decreasing on $(0, \sqrt{3})$ and $(-\infty, 0)$

7. Sketch the Graph

- Asymptotes:

- Vertical asymptote at $x = 0$

- Horizontal asymptote at $y = 1$

- Intercepts:

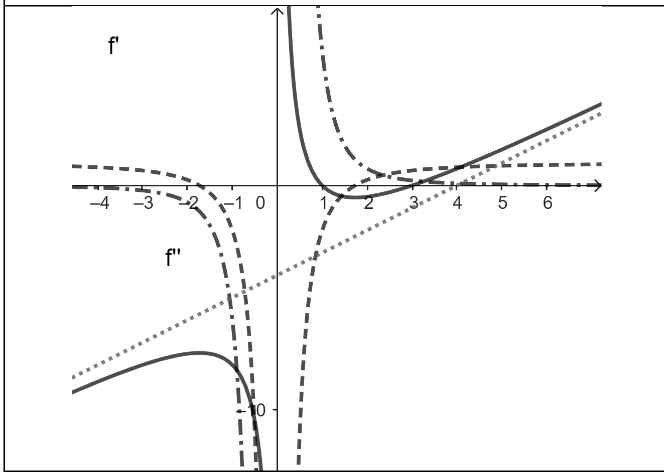
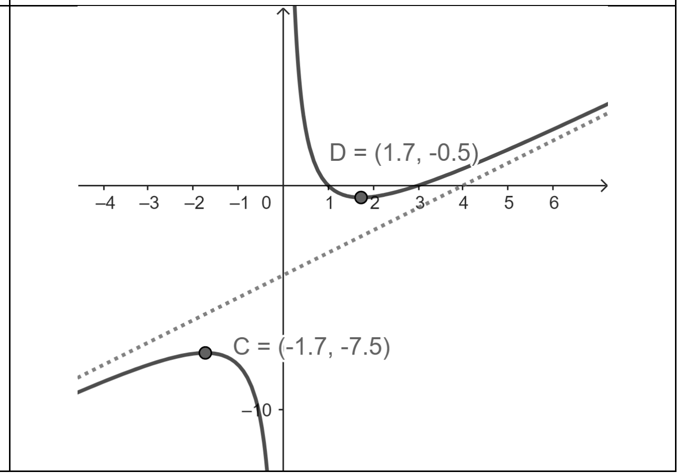
- x-intercepts at $x = 1$ and $x = 3$

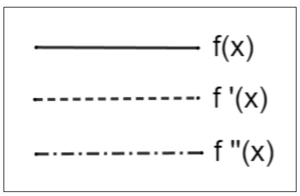
- Critical Points:

- Local Minimum at $x = \sqrt{3}$: $y = -0.5$

- Local Maximum at $x = -\sqrt{3}$: $y = -7.5$

$f(x) = \frac{x^2 - 3x - 4}{x}$							
$f'(x) = 1 - \frac{3}{x^2}$	+	$x = \sqrt{3}$ (local min)	-	0 (undefined)	-	$x = -\sqrt{3}$ (local max)	+
$f''(x) = \frac{6}{x^3}$	-	-	-	0 (undefined)	+	+	+

$f'(x)=1-\frac{3}{x^2}$, $f''(x)=\frac{6}{x^3}$	$f(x)=\frac{x^2-3x-4}{x}$
	

Legend: 

0-7. Derivative Formula

1. Derivative and Integral Rules

	Derivative	Integral (Antiderivative)
1	$\frac{d}{dx}n=0$	$\int 0dx=C$
2	$\frac{d}{dx}x=1$	$\int 1dx=x+C$
3	$\frac{d}{dx}\left[x^n\right]=nx^{n-1}$	$\int \left[x^n\right]dx=\frac{x^{n+1}}{n+1}+C$
4	$\frac{d}{dx}\left[e^x\right]=e^x$	$\int \left[e^x\right]dx=e^x+C$
5	$\frac{d}{dx}\left[\ln x\right]=\frac{1}{x}$	$\int \left[\frac{1}{x}\right]dx=\ln x+C$
6	$\frac{d}{dx}\left[n^x\right]=n^x\ln n$	$\int \left[n^x\right]dx=\frac{n^x}{\ln n}+C$
7	$\frac{d}{dx}\left[\sin x\right]=\cos x$	$\int \left[\cos x\right]dx=\sin x+C$
8	$\frac{d}{dx}\left[\cos x\right]=-\sin x$	$\int \left[\sin x\right]dx=-\cos x+C$
9	$\frac{d}{dx}\left[\tan x\right]=\sec^2 x$	$\int \left[\sec^2 x\right]dx=\tan x+C$
10	$\frac{d}{dx}\left[\cot x\right]=-\csc^2 x$	$\int \left[\csc^2 x\right]dx=-\cot x+C$
11	$\frac{d}{dx}\left[\sec x\right]=\sec x\tan x$	$\int \left[\tan x\sec x\right]dx=\sec x+C$
12	$\frac{d}{dx}\left[\csc x\right]=-\csc x\cot x$	$\int \left[\cot x\csc x\right]dx=-\csc x+C$
13	$\frac{d}{dx}\left[\arcsin x\right]=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx=\arcsin x+C$
14	$\frac{d}{dx}\left[\arccos x\right]=-\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx=\arccos x+C$
15	$\frac{d}{dx}\left[\arctan x\right]=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx=\arctan x+C$

16	$\frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \operatorname{arccot} x + C$
18	$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C$
19	$\frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \operatorname{arccsc} x + C$

2. General Differentiation Rules

Let **c** be a real number, **n** be a rational number, **u** and **v** be differentiable functions of **x**, let **f** be a differentiable function of **u**, and let **a** be a positive real number ($a \neq 1$).

	Rules	
1	Constant Rule	$\frac{d}{dx}[c] = 0$
2	Constant Multiple Rule	$\frac{d}{dx}[cu] = cu'$
3	Product Rule	$\frac{d}{dx}[uv] = uv' + vu'$
4	Chain Rule	$\frac{d}{dx}[f(u)] = f'(u)u'$
5	(Simple) Power Rule	$\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$
6	Sum or Difference Rule	$\frac{d}{dx}[u \pm v] = u' \pm v'$
7	Quotient Rule	$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
8	General Power Rule	$\frac{d}{dx}[u^n] = nu^{n-1}u'$
9	Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$ $\frac{d}{dx}[\cos x] = -\sin x$ $\frac{d}{dx}[\tan x] = \sec^2 x$

		$\frac{d}{dx}[\cot x] = -\csc^2 x$ $\frac{d}{dx}[\sec x] = \sec x \tan x$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$
10	Derivatives of Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\sin u] = (\cos u)u'$ $\frac{d}{dx}[\cos u] = -(\sin u)u'$ $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$ $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$ $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$ $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
11	Derivatives of Inverse Trigonometric Functions	$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arccot}\, x] = -\frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{x\sqrt{x^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, x] = -\frac{1}{x\sqrt{x^2-1}}$
12	Derivatives of Inverse Trigonometric Functions (**u** be differentiable functions of **x)**	$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$ $\frac{d}{dx}[\text{arccot}\, u] = \frac{-u'}{1+u^2}$ $\frac{d}{dx}[\text{arcsec}\, u] = \frac{u'}{\|u\|\sqrt{u^2-1}}$

		$\frac{d}{dx}[\text{arccsc}u]=\frac{-u'}{\|u\|\sqrt{u^2-1}}$
13	Derivatives of Basic Hyperbolic Trigonometric Functions $\sinh(x)=\frac{e^x-e^{-x}}{2}$ $\cosh(x)=\frac{e^x+e^{-x}}{2}$	$\frac{d}{dx}[\sinh(x)]=\cosh(x)$ $\frac{d}{dx}[\cosh(x)]=\sinh(x)$ $\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$ $\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$ $\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$ $\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$
14	Derivatives of Inverse Hyperbolic Trigonometric Functions	$\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$ $\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$ $\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$ $\frac{d}{dx}[\text{sech}^{-1}(x)]=\frac{-1}{x\sqrt{1-x^2}}$ $\frac{d}{dx}[\text{csch}^{-1}(x)]=\frac{-1}{\|x\|\sqrt{1+x^2}}$ $\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$
15	Derivatives of Exponential and Logarithmic Functions	$\frac{d}{dx}[e^x]=e^x$ $\frac{d}{dx}[\ln x]=\frac{1}{x}$ $\frac{d}{dx}[a^x]=(\ln a)a^x$ $\frac{d}{dx}[\log_a x]=\frac{1}{(\ln a)x}$
16	Basic Differentiation Rules for Elementary Functions (**u & v** be differentiable functions of **x)**	$\frac{d}{dx}[u^n]=nu^{n-1}u'$ $\frac{d}{dx}[\|u\|]=\frac{u}{\|u\|}u',\quad u\neq 0$ $\frac{d}{dx}[\ln u]=\frac{u'}{u}$

		$\frac{d}{dx}[e^u] = e^u u'$ $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$ $\frac{d}{dx}[a^u] = (\ln a)a^u u'$

3. Hyperbolic functions are analogs of the circular trigonometric functions, but for a hyperbola. They are extensively used in various areas of mathematics, including algebra, calculus, and complex analysis. Here are the basic hyperbolic functions along with their definitions:

1	Hyperbolic Sine ($\sinh x$)	$\sinh x = \frac{e^x - e^{-x}}{2}$
2	Hyperbolic Cosine ($\cosh x$)	$\cosh x = \frac{e^x + e^{-x}}{2}$
3	Hyperbolic Tangent ($\tanh x$)	$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4	Hyperbolic Cosecant ($\text{csch}\, x$)	$\text{csch}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$
5	Hyperbolic Secant ($\text{sech}\, x$)	$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
6	Hyperbolic Cotangent ($\coth x$)	$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

4. List of antiderivative formulas covering a wider range of functions. These include basic functions, exponential and logarithmic functions, trigonometric functions, and some of their inverses

	Functions	**Antiderivative formulas**
1	Constant Function	$\int a\,dx = ax + C$
2	Power Function	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3	Exponential Function	$\int e^x\,dx = e^x + C$

4	General Exponential Function	$\int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (a > 0, a \neq 1)$
5	Natural Logarithm	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
6	Sine Functions	$\int \sin(x)dx = -\cos(x) + C$
7	Cosine Functions	$\int \cos(x)dx = \sin(x) + C$
8	Tangent Functions	$\int \tan(x)dx = -\ln\lvert\cos(x)\rvert + C$
9	Cotangent (cot) Functions	$\int \cot(x)dx = \ln\lvert\sin(x)\rvert + C$
10	Secant (sec) Functions	$\int \sec(x)dx = \ln\lvert\sec(x) + \tan(x)\rvert + C$
11	Cosecant (csc) Functions	$\int \csc(x)dx = -\ln\lvert\csc(x) + \cot(x)\rvert + C$
12	Inverse Sine (arcsin) Functions	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1}(x) + C$
13	Inverse Tangent (arctan) Functions	$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + C$
14	sinh (Hyperbolic Sine) Functions	$\int \sinh(x)dx = \cosh(x) + C$
15	cosh (Hyperbolic Cosine) Functions	$\int \cosh(x)dx = \sinh(x) + C$
16	Integral of sec^2	$\int \sec^2(x)dx = \tan(x) + C$
17	Integral of csc^2	$\int \csc^2(x)dx = -\cot(x) + C$

0-8. Find Derivatives

1) Find the derivative of the function $f(x) = 7$. (Constant Rule)	2) Find the derivative of the function $f(x) = 5x^3$. (Constant Multiple Rule)
3) Find the derivative of the function $f(x) = x^2 \sin(x)$. (Product Rule)	4) Find the derivative of the function $f(x) = \sin(3x)$. (Chain Rule)
5) Find the derivative of the function $f(x) = x^5$. ((Simple) Power Rule)	6) Find the derivative of the function $f(x) = x^3 - 4x + 7$. (Sum or Difference Rule)
7) Find the derivative of the function $f(x) = \dfrac{x^2}{\sin(x)}$. (Quotient Rule)	8) Find the derivative of the function $f(x) = (3x^2 + 2)^4$. (General Power Rule)
9) Find the derivative of the function $f(x) = \tan(x)$. (Derivatives of Trigonometric Functions)	10) Find the derivative of the function $f(x) = \sin(x)$. (Derivative of $\sin(x)$) -
11) Find the derivative of the function $f(x) = \cos(x)$. (Derivative of $\cos(x)$)	12) Find the derivative of the function $f(x) = \tan(2x)$. (Derivative of $\tan(x)$)
13) Find the derivative of the function $f(x) = \cot(x)$. (Derivative of $\cot(x)$)	14) Find the derivative of the function $f(x) = \sec(x)$. (Derivative of $\sec(x)$)

15) Find the derivative of the function $f(x) = \csc(x)$. (Derivative of $\csc(x)$)	16) Find the derivative of the function $f(x) = \sin(3x^2 + 2x)$. (Derivative of $\sin(u)$ where u is a function of x)
17) Find the derivative of the function $f(x) = \cos(x^3 - x)$. (Derivative of $\cos(u)$ where u is a function of x)	18) Find the derivative of the function $f(x) = \tan(2x^2 - 3x)$. (Derivative of $\tan(u)$ where u is a function of x)
19) Find the derivative of the function $f(x) = \cot(4x^3 + x^2)$. (Derivative of $\cot(u)$ where u is a function of x)	20) Find the derivative of the function $f(x) = \sec(3x^2 + x)$. (Derivative of $\sec(u)$ where u is a function of x)
21) Find the derivative of the function $f(x) = \csc(x^2 + 2x)$. (Derivative of $\csc(u)$ where u is a function of x)	22) Find the derivative of the function $f(x) = \sinh(x)$. (Derivative of $\sinh(x)$)
23) Find the derivative of the function $f(x) = \cosh(x)$. (Derivative of $\cosh(x)$)	24) Find the derivative of the function $f(x) = \tanh(x)$. (Derivative of $\tanh(x)$)
25) Find the derivative of the function $f(x) = \operatorname{sech}(x)$. (Derivative of $\operatorname{sech}(x)$)	26) Find the derivative of the function $f(x) = \operatorname{csch}(x)$. (Derivative of $\operatorname{csch}(x)$)
27) Find the derivative of the function $f(x) = \coth(x)$. (Derivative of $\coth(x)$)	28) Find the derivative of the function $f(x) = \sinh^{-1}(x)$. (Derivative of $\sinh^{-1}(x)$)
29) Find the derivative of the function $f(x) = \cosh^{-1}(x)$. (Derivative of $\cosh^{-1}(x)$)	30) Find the derivative of the function $f(x) = \tanh^{-1}(x)$. (Derivative of $\tanh^{-1}(x)$)

31) Find the derivative of the function $f(x) = \text{sech}^{-1}(x)$. (Derivative of $\text{sech}^{-1}(x)$)	32) Find the derivative of the function $f(x) = \text{csch}^{-1}(x)$. (Derivative of $\text{csch}^{-1}(x)$)
33) Find the derivative of the function $f(x) = \coth^{-1}(x)$. (Derivative of $\coth^{-1}(x)$)	34) Find the derivative of the function $f(x) = e^x$. (Derivative of e^x)
35) Find the derivative of the function $f(x) = \ln(x)$. (Derivative of $\ln(x)$)	36) Find the derivative of the function $f(x) = 2^x$. (Derivative of a^x)
37) Find the derivative of the function $f(x) = \log_2(x)$. (Derivative of $\log_a(x)$)	38) Find the derivative of the function $f(x) = (3x^2 + 2)^5$. (Derivative of u^n)
39) Find the derivative of the function $f(x) = \lvert 3x - 4 \rvert$. (Derivative of $\lvert u \rvert$)	40) Find the derivative of the function $f(x) = \ln(2x^3 + 5)$. (Derivative of $\ln(u)$)
41) Find the derivative of the function $f(x) = e^{4x^2}$. (Derivative of e^u)	42) Find the derivative of the function $f(x) = \log_3(x^2 + 1)$. (Derivative of $\log_a(u)$)
43) Find the derivative of the function $f(x) = 5^{3x}$. (Derivative of a^u)	

Solutions:

1) Find the derivative of the function $f(x)=7$ **. (Constant Rule)** - Using the constant rule, which states $\frac{d}{dx}[c]=0$ - $f'(x)=\frac{d}{dx}[7]=0$	**2) Find the derivative of the function** $f(x)=5x^3$ **. (Constant Multiple Rule)** - Using the constant multiple rule, which states $\frac{d}{dx}[cu]=cu'$ - $f'(x)=\frac{d}{dx}[5x^3]=5\cdot\frac{d}{dx}[x^3]=5\cdot 3x^2=15x^2$
3) Find the derivative of the function $f(x)=x^2\sin(x)$ **. (Product Rule)** - Using the product rule, which states $\frac{d}{dx}[uv]=uv'+vu'$ - $u=x^2$, $v=\sin(x)$ - $u'=2x$, $v'=\cos(x)$ - $f'(x)=(x^2)'\sin(x)+x^2(\sin(x))'$ $=2x\sin(x)+x^2\cos(x)$	**4) Find the derivative of the function** $f(x)=\sin(3x)$ **. (Chain Rule)** - Using the chain rule, which states $\frac{d}{dx}[f(u)]=f'(u)u'$ - $f(u)=\sin(u)$, $u=3x$ - $f'(u)=\cos(u)$, $u'=3$ - $f'(x)=\cos(3x)\cdot 3=3\cos(3x)$
5) Find the derivative of the function $f(x)=x^5$ **. ((Simple) Power Rule)** - Using the power rule, which states $\frac{d}{dx}[x^n]=nx^{n-1}$ - $f'(x)=\frac{d}{dx}[x^5]=5x^4$	**6) Find the derivative of the function** $f(x)=x^3-4x+7$ **. (Sum or Difference Rule)** - Using the sum or difference rule, which states $\frac{d}{dx}[u\pm v]=u'\pm v'$ - $f'(x)=\frac{d}{dx}[x^3]-\frac{d}{dx}[-4x]+\frac{d}{dx}[7]$ $=3x^2-4+0=3x^2-4$
7) Find the derivative of the function $f(x)=\frac{x^2}{\sin(x)}$ **. (Quotient Rule)** - Using the quotient rule, which states $\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{vu'-uv'}{v^2}$ - $u=x^2$, $v=\sin(x)$ - $u'=2x$, $v'=\cos(x)$	**8) Find the derivative of the function** $f(x)=(3x^2+2)^4$ **. (General Power Rule)** - Using the general power rule, which states $\frac{d}{dx}[u^n]=nu^{n-1}u'$ - $u=3x^2+2$, $u'=6x$ - $n=4$ - $f'(x)=4(3x^2+2)^3\cdot 6x=24x(3x^2+2)^3$

- $f'(x) = \dfrac{\sin(x)\cdot 2x - x^2 \cdot \cos(x)}{\sin^2(x)}$ $= \dfrac{2x\sin(x) - x^2\cos(x)}{\sin^2(x)}$	
9) Find the derivative of the function $f(x) = \tan(x)$ **. (Derivatives of Trigonometric Functions)** - Using the derivative rule for the tangent function, which states $\dfrac{d}{dx}[\tan(x)] = \sec^2(x)$ - $f'(x) = \dfrac{d}{dx}[\tan(x)] = \sec^2(x)$	**10) Find the derivative of the function** $f(x) = \sin(x)$ **. (Derivative of** $\sin(x)$ **)** - Using the derivative rule for the sine function, which states $\dfrac{d}{dx}[\sin(x)] = \cos(x)$ - $f'(x) = \dfrac{d}{dx}[\sin(x)] = \cos(x)$
11) Find the derivative of the function $f(x) = \cos(x)$ **. (Derivative of** $\cos(x)$ **)** - Using the derivative rule for the cosine function, which states $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$ - $f'(x) = \dfrac{d}{dx}[\cos(x)] = -\sin(x)$	**12) Find the derivative of the function** $f(x) = \tan(2x)$ **. (Derivative of** $\tan(x)$ **)** - Using the derivative rule for the tangent function, which states $\dfrac{d}{dx}[\tan(u)] = \sec^2(u)\cdot u'$ - $f'(x) = \dfrac{d}{dx}[\tan(2x)] = \sec^2(2x)\cdot 2$
13) Find the derivative of the function $f(x) = \cot(x)$ **. (Derivative of** $\cot(x)$ **)** - Using the derivative rule for the cotangent function, which states $\dfrac{d}{dx}[\cot(x)] = -\csc^2(x)$ - $f'(x) = \dfrac{d}{dx}[\cot(x)] = -\csc^2(x)$	**14) Find the derivative of the function** $f(x) = \sec(x)$ **. (Derivative of** $\sec(x)$ **)** - Using the derivative rule for the secant function, which states $\dfrac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$ - $f'(x) = \dfrac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$
15) Find the derivative of the function $f(x) = \csc(x)$ **. (Derivative of** $\csc(x)$ **)** - Using the derivative rule for the cosecant function, which states $\dfrac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$ - $f'(x) = \dfrac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$	**16) Find the derivative of the function** $f(x) = \sin(3x^2 + 2x)$ **. (Derivative of** $\sin(u)$ **where u is a function of x)** - Using the chain rule and the derivative of sine, which states $\dfrac{d}{dx}[\sin(u)] = (\cos(u))u'$ - $u = 3x^2 + 2x$, $u' = 6x + 2$ - $f'(x) = \cos(3x^2 + 2x)\cdot(6x + 2)$ $= (6x + 2)\cos(3x^2 + 2x)$

17) Find the derivative of the function $f(x)=\cos(x^3-x)$ **. (Derivative of** $\cos(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of cosine, which states $\frac{d}{dx}[\cos(u)]=-(\sin(u))u'$
- $u=x^3-x$, $u'=3x^2-1$
- $f'(x)=-\sin(x^3-x)\cdot(3x^2-1)$
  $=-(3x^2-1)\sin(x^3-x)$

18) Find the derivative of the function $f(x)=\tan(2x^2-3x)$ **. (Derivative of** $\tan(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of tangent, which states $\frac{d}{dx}[\tan(u)]=(\sec^2(u))u'$
- $u=2x^2-3x$, $u'=4x-3$
- $f'(x)=\sec^2(2x^2-3x)\cdot(4x-3)$
  $=(4x-3)\sec^2(2x^2-3x)$

19) Find the derivative of the function $f(x)=\cot(4x^3+x^2)$ **. (Derivative of** $\cot(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of cotangent, which states
  $\frac{d}{dx}[\cot(u)]=-(\csc^2(u))u'$
- $u=4x^3+x^2$, $u'=12x^2+2x$
- $f'(x)=-\csc^2(4x^3+x^2)\cdot(12x^2+2x)$
  $=-(12x^2+2x)\csc^2(4x^3+x^2)$

20) Find the derivative of the function $f(x)=\sec(3x^2+x)$ **. (Derivative of** $\sec(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of secant, which states
  $\frac{d}{dx}[\sec(u)]=(\sec(u)\tan(u))u'$
- $u=3x^2+x$, $u'=6x+1$
- $f'(x)=\sec(3x^2+x)\tan(3x^2+x)\cdot(6x+1)$
  $=(6x+1)\sec(3x^2+x)\tan(3x^2+x)$

21) Find the derivative of the function $f(x)=\csc(x^2+2x)$ **. (Derivative of** $\csc(u)$ **where u is a function of x)**

- Using the chain rule and the derivative of cosecant, which states
  $\frac{d}{dx}[\csc(u)]=-(\csc(u)\cot(u))u'$
- $u=x^2+2x$, $u'=2x+2$
- $f'(x)=-\csc(x^2+2x)\cot(x^2+2x)\cdot(2x+2)$
  $=-(2x+2)\csc(x^2+2x)\cot(x^2+2x)$

22) Find the derivative of the function $f(x)=\sinh(x)$ **. (Derivative of** $\sinh(x)$ **)**

- Using the derivative rule for the hyperbolic sine function, which states $\frac{d}{dx}[\sinh(x)]=\cosh(x)$
- $f'(x)=\frac{d}{dx}[\sinh(x)]=\cosh(x)$

23) Find the derivative of the function $f(x)=\cosh(x)$ **. (Derivative of** $\cosh(x)$ **)**

- Using the derivative rule for the hyperbolic cosine function, which states
  $\frac{d}{dx}[\cosh(x)]=\sinh(x)$

24) Find the derivative of the function $f(x)=\tanh(x)$ **. (Derivative of** $\tanh(x)$ **)**

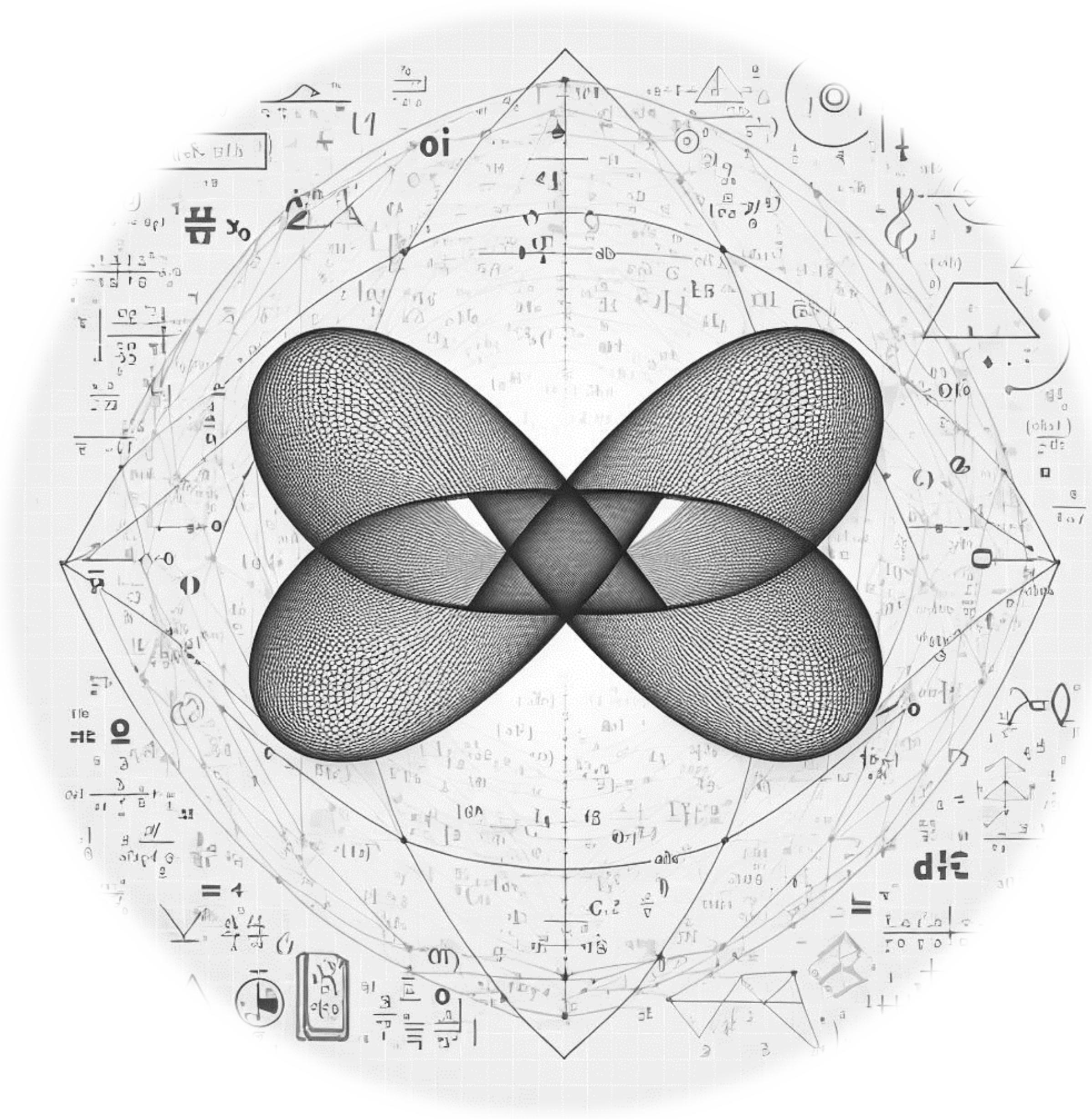
- Using the derivative rule for the hyperbolic tangent function, which states
  $\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$

- $f'(x)=\frac{d}{dx}[\cosh(x)]=\sinh(x)$	- $f'(x)=\frac{d}{dx}[\tanh(x)]=\text{sech}^2(x)$
25) Find the derivative of the function $f(x)=\text{sech}(x)$ **. (Derivative of** $\text{sech}(x)$ **)** - Using the derivative rule for the hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$ - $f'(x)=\frac{d}{dx}[\text{sech}(x)]=-\text{sech}(x)\tanh(x)$	**26) Find the derivative of the function** $f(x)=\text{csch}(x)$ **. (Derivative of** $\text{csch}(x)$ **)** - Using the derivative rule for the hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$ - $f'(x)=\frac{d}{dx}[\text{csch}(x)]=-\text{csch}(x)\coth(x)$
27) Find the derivative of the function $f(x)=\coth(x)$ **. (Derivative of** $\coth(x)$ **)** - Using the derivative rule for the hyperbolic cotangent function, which states $\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$ - $f'(x)=\frac{d}{dx}[\coth(x)]=-\text{csch}^2(x)$	**28) Find the derivative of the function** $f(x)=\sinh^{-1}(x)$ **. (Derivative of** $\sinh^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic sine function, which states $\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$ - $f'(x)=\frac{d}{dx}[\sinh^{-1}(x)]=\frac{1}{\sqrt{x^2+1}}$
29) Find the derivative of the function $f(x)=\cosh^{-1}(x)$ **. (Derivative of** $\cosh^{-1}(x)$ **):** - Using the derivative rule for the inverse hyperbolic cosine function, which states $\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$ - $f'(x)=\frac{d}{dx}[\cosh^{-1}(x)]=\frac{1}{\sqrt{x^2-1}}$	**30) Find the derivative of the function** $f(x)=\tanh^{-1}(x)$ **. (Derivative of** $\tanh^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic tangent function, which states $\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$ - $f'(x)=\frac{d}{dx}[\tanh^{-1}(x)]=\frac{1}{1-x^2}$
31) Find the derivative of the function $f(x)=\text{sech}^{-1}(x)$ **. (Derivative of** $\text{sech}^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic secant function, which states $\frac{d}{dx}[\text{sech}^{-1}(x)]=\frac{-1}{x\sqrt{1-x^2}}$ - $f'(x)=\frac{d}{dx}[\text{sech}^{-1}(x)]=\frac{-1}{x\sqrt{1-x^2}}$	**32) Find the derivative of the function** $f(x)=\text{csch}^{-1}(x)$ **. (Derivative of** $\text{csch}^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic cosecant function, which states $\frac{d}{dx}[\text{csch}^{-1}(x)]=\frac{-1}{\|x\|\sqrt{1+x^2}}$ - $f'(x)=\frac{d}{dx}[\text{csch}^{-1}(x)]=\frac{-1}{\|x\|\sqrt{1+x^2}}$

33) Find the derivative of the function $f(x)=\coth^{-1}(x)$ **. (Derivative of** $\coth^{-1}(x)$ **)** - Using the derivative rule for the inverse hyperbolic cotangent function, which states $\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$ - $f'(x)=\frac{d}{dx}[\coth^{-1}(x)]=\frac{1}{1-x^2}$	**34) Find the derivative of the function** $f(x)=e^x$ **. (Derivative of** e^x **)** - Using the derivative rule for the exponential function, which states $\frac{d}{dx}[e^x]=e^x$ - $f'(x)=\frac{d}{dx}[e^x]=e^x$
35) Find the derivative of the function $f(x)=\ln(x)$ **. (Derivative of** $\ln(x)$ **)** - Using the derivative rule for the natural logarithm function, which states $\frac{d}{dx}[\ln(x)]=\frac{1}{x}$ - $f'(x)=\frac{d}{dx}[\ln(x)]=\frac{1}{x}$	**36) Find the derivative of the function** $f(x)=2^x$ **. (Derivative of** a^x **)** - Using the derivative rule for the exponential function with base a, which states $\frac{d}{dx}[a^x]=(\ln a)a^x$ - $f'(x)=\frac{d}{dx}[2^x]=(\ln 2)2^x$
37) Find the derivative of the function $f(x)=\log_2(x)$ **. (Derivative of** $\log_a(x)$ **)** - Using the derivative rule for the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(x)]=\frac{1}{(\ln a)x}$ - $f'(x)=\frac{d}{dx}[\log_2(x)]=\frac{1}{(\ln 2)x}$	**38) Find the derivative of the function** $f(x)=(3x^2+2)^5$ **. (Derivative of** u^n **)** - Using the general power rule, which states $\frac{d}{dx}[u^n]=nu^{n-1}u'$ - $u=3x^2+2$, $u'=6x$ - $f'(x)=5(3x^2+2)^4\cdot 6x=30x(3x^2+2)^4$
39) Find the derivative of the function $f(x)=\|3x-4\|$ **. (Derivative of |u|)** - Using the rule for the derivative of the absolute value function, which states $\frac{d}{dx}[\|u\|]=\frac{u}{\|u\|}u'$ where $u\neq 0$ - $u=3x-4$, $u'=3$ - $f'(x)=\frac{3x-4}{\|3x-4\|}\cdot 3=\frac{3(3x-4)}{\|3x-4\|}$	**40) Find the derivative of the function** $f(x)=\ln(2x^3+5)$ **. (Derivative of** $\ln(u)$ **)** - Using the rule for the derivative of the natural logarithm function, which states $\frac{d}{dx}[\ln(u)]=\frac{u'}{u}$ - $u=2x^3+5$, $u'=6x^2$ - $f'(x)=\frac{6x^2}{2x^3+5}$
41) Find the derivative of the function $f(x)=e^{4x^2}$ **. (Derivative of** e^u **)**	**42) Find the derivative of the function** $f(x)=\log_3(x^2+1)$ **. (Derivative of** $\log_a(u)$ **)**

- Using the rule for the derivative of the exponential function, which states $\frac{d}{dx}[e^u] = e^u u'$ - $u = 4x^2$, $u' = 8x$ - $f'(x) = e^{4x^2} \cdot 8x = 8xe^{4x^2}$	- Using the rule for the derivative of the logarithmic function with base a, which states $\frac{d}{dx}[\log_a(u)] = \frac{u'}{(\ln a)u}$ - $u = x^2 + 1$, $u' = 2x$ - $f'(x) = \frac{2x}{(\ln 3)(x^2+1)} = \frac{2x}{(\ln 3)(x^2+1)}$
43) Find the derivative of the function $f(x) = 5^{3x}$ **. (Derivative of** a^u **)** - Using the rule for the derivative of the exponential function with base a, which states $\frac{d}{dx}[a^u] = (\ln a)a^u u'$ - $u = 3x$, $u' = 3$ - $f'(x) = (\ln 5)5^{3x} \cdot 3 = 3(\ln 5)5^{3x}$	

Chapter 1. Limits and Continuity for Point Limits



1-1. Introduction to Limits (Point Limits)

The **concept of limits** is foundational in calculus and involves approaching a particular point on the function.

A limit describes the behavior of a function as it approaches a certain value (x-value), regardless of what the function's value is exactly at that point.

This concept is crucial for dealing with situations where the function becomes difficult or impossible to evaluate directly at that point due to discontinuities or undefined expressions.

A **point limit** is specifically the value that a function approaches as the input (or x-value) approaches a particular point. It's expressed as:

$$\lim_{x \to a} f(x) = L$$

This notation means that as x gets closer and closer to a, $f(x)$ gets arbitrarily close to L.

Examples:

1) Evaluate the limit: $\lim_{x \to 3}(2x+1)$.	2) Compute the limit: $\lim_{x \to -2}(x^2-4)$.
3) Find the limit:. $\lim_{x \to 0}\frac{x^2-1}{x-1}$..	4) Determine the limit: $\lim_{x \to 4}\frac{3x-12}{x-4}$.

5) Calculate the limit: $\lim_{x\to 2}\frac{x^2-x-2}{x^2-4}$.	6) Find the limit, if it exists: $\lim_{x\to 0}\frac{\sin(x)}{x}$.
7) Find the limit, if it exists: $\lim_{x\to 0}\frac{\sin(2x)}{x}$.	8) Evaluate the limit: $\lim_{x\to 0}\frac{e^x-1}{x}$.
9) Determine the limit: $\lim_{x\to \infty}\frac{2x^2+3x}{5x^2-2}$.	10) Find the limit: $\lim_{x\to 0}\frac{\tan(x)}{x}$.

Solutions:

1) Evaluate the limit: $\lim_{x\to 3}(2x+1)$ - To find the limit as x approaches 3, substitute 3 into the function: $2(3)+1=6+1=7$ - Therefore, $\lim_{x\to 3}(2x+1)=7$.	**2) Compute the limit:** $\lim_{x\to -2}(x^2-4)$**.** - Substitute -2 into the function: $(-2)^2-4=4-4=0$ - So, $\lim_{x\to -2}(x^2-4)=0$.
3) Find the limit: $\lim_{x\to 0}\frac{x^2-1}{x-1}$**.** - Factoring the numerator, we get: $\lim_{x\to 0}\frac{(x+1)(x-1)}{x-1}$ - The $x-1$ terms cancel out, giving: $\lim_{x\to 0}(x+1)=0+1=1$	**4) Determine the limit:** $\lim_{x\to 4}\frac{3x-12}{x-4}$**.** - Factoring out a 3 from the numerator, we get: $\lim_{x\to 4}\frac{3(x-4)}{x-4}$ - Canceling out the $x-4$ terms gives us: $\lim_{x\to 4}3=3$
5) Calculate the limit: $\lim_{x\to 2}\frac{x^2-x-2}{x^2-4}$**.** - Factoring both the numerator and the denominator: $\lim_{x\to 2}\frac{(x-2)(x+1)}{(x+2)(x-2)}$ - Canceling the $x-2$ terms, we have: $\lim_{x\to 2}\frac{x+1}{x+2}=\frac{2+1}{2+2}=\frac{3}{4}$	**6) Find the limit, if it exists:** $\lim_{x\to 0}\frac{\sin(x)}{x}$**.** - This is a standard limit in calculus, and it is known that: $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ (Memorize this for now)
7) Find the limit, if it exists: $\lim_{x\to 0}\frac{\sin(2x)}{x}$**.** - This is a standard limit in calculus, and it is known that: $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ (Memorize this for now) - Substitute $2x=t$: $\lim_{2x\to 0}\frac{\sin(2x)}{2x}=\lim_{t\to 0}\frac{\sin(t)}{t}$ - Solve: $\lim_{x\to 0}\frac{\sin(2x)}{2x}\times\frac{2}{1}=1\times 2=2$	**8) Evaluate the limit:** $\lim_{x\to 0}\frac{e^x-1}{x}$**.** - This is another standard limit: - $\lim_{x\to 0}\frac{e^x-1}{x}=1$ (Memorize this)

9) Determine the limit:	10) Find the limit:
9) Determine the limit: $\lim_{x\to\infty}\frac{2x^2+3x}{5x^2-2}$. - Divide every term by x^2 (The highest power): $\lim_{x\to\infty}\frac{2+\frac{3}{x}}{5-\frac{2}{x^2}}=\frac{2}{5}$ - As $x\to\infty$, $\frac{3}{x}\to 0$ and $\frac{2}{x^2}\to 0$. - So, the expression simplifies to: $\frac{2+0}{5-0}=\frac{2}{5}$	**10) Find the limit:** $\lim_{x\to 0}\frac{tan(x)}{x}$. - Use $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ and $\tan(x)=\frac{\sin(x)}{\cos(x)}$ - $\lim_{x\to 0}\frac{tan(x)}{x}=\lim_{x\to 0}\frac{\sin(x)}{\cos(x)}\times\frac{1}{x}$ $=\lim_{x\to 0}\frac{\sin(x)}{x}\times\frac{1}{\cos(x)}$ $=\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)\times\left(\lim_{x\to 0}\frac{1}{\cos(x)}\right)$ $=(1)\times\left(\frac{1}{\cos(0)}\right)=1\times\frac{1}{1}=1$ - So, $\lim_{x\to 0}\frac{tan(x)}{x}=1$

1-2. Evaluating Limits Using Direct Substitution

Direct substitution is a straightforward method for evaluating limits in calculus.

It involves substituting the point at which the limit is taken directly into the function, assuming the function is continuous at that point.

This method is particularly useful and efficient when the function **does not exhibit (DNE)** any indeterminate forms like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Examples:

1) Evaluate $\lim_{x \to 5}(3x-7)$.	2) Compute $\lim_{x \to -1}(x^3+2x^2-x+1)$.
3) Find the limit: $\lim_{x \to 2}\frac{x^2-4}{x-2}$.	4) Evaluate $\lim_{x \to -3}\frac{x+3}{x^2+9}$.
5) Evaluate the limit: $\lim_{x \to 0}\frac{\sqrt{x+4}-2}{x}$.	6) Determine $\lim_{x \to \infty}\frac{\sqrt{x^2+x}-x}{x}$.

Solutions:

1) Evaluate $\lim_{x\to 5}(3x-7)$. - Directly substitute $x=5$: $3(5)-7=15-7=8$ - So, $\lim_{x\to 5}(3x-7)=8$.	**2) Compute** $\lim_{x\to -1}(x^3+2x^2-x+1)$. - Substitute $x=-1$: $(-1)^3+2(-1)^2-(-1)+1=-1+2+1+1=3$ - Thus, $\lim_{x\to -1}(x^3+2x^2-x+1)=3$.
3) Find the limit: $\lim_{x\to 2}\frac{x^2-4}{x-2}$. - Direct substitution would lead to a $0/0$ indeterminate form, which suggests a need for factoring: $x^2-4=(x-2)(x+2)$ - Then: $\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$ - Now we can directly substitute since $x+2$ is continuous at $x=2$: $2+2=4$ - So, $\lim_{x\to 2}\frac{x^2-4}{x-2}=4$.	**4) Evaluate** $\lim_{x\to -3}\frac{x+3}{x^2+9}$. - Direct substitution yields: $\frac{-3+3}{(-3)^2+9}=\frac{0}{18}=0$ - Therefore, $\lim_{x\to -3}\frac{x+3}{x^2+9}=0$.
5) Evaluate the limit: $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$. - Direct substitution gives which is indeterminate $0/0$. To resolve this, rationalize the numerator: $\lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}$ - This simplifies to: $\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}$ $=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}$ - Now direct substitution yields: $\frac{1}{\sqrt{0+4}+2}=\frac{1}{4}$	**6) Determine** $\lim_{x\to\infty}\frac{\sqrt{x^2+x}-x}{x}$. - At infinity, direct substitution does not work directly, and we need to manipulate the expression. - Factor out x from the square root and simplify: $=\lim_{x\to\infty}\frac{\sqrt{x^2(1+\frac{1}{x})}-x}{x}$ $=\lim_{x\to\infty}\frac{x\sqrt{1+\frac{1}{x}}-x}{x}$ $=\lim_{x\to\infty}(\sqrt{1+\frac{1}{x}}-1)$ - Direct substitution now yields: as $x\to\infty$, $\frac{1}{x}\to 0$ - $\sqrt{1+0}-1=0$.

1-3. Simplifying Functions Through Factoring for Substitution

Simplifying functions through factoring is a powerful technique used in calculus to make limit evaluations more straightforward, especially when direct substitution results in an indeterminate form like $\frac{0}{0}$.

Factoring allows us to cancel out terms in the numerator and denominator, which may prevent direct evaluation but become resolvable after simplification.

Examples:

1) Evaluate $\lim_{x\to 5}\frac{x^2-25}{x-5}$.	2) Find the limit $\lim_{x\to -3}\frac{x^2-9}{x+3}$.
3) Compute $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$.	4) Determine $\lim_{x\to 1}\frac{x^2-1}{x^2-x}$.
5) Evaluate $\lim_{x\to 0}\frac{x^2-4x}{x}$.	6) Calculate $\lim_{x\to 3}\frac{x^3-27}{x-3}$.

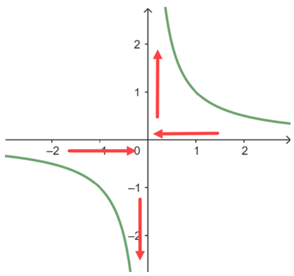
7) Determine $\lim_{x\to -2}\frac{x^2-2x-8}{x+2}$.	8) Evaluate $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$.
9) Calculate $\lim_{x\to 4}\frac{x^2-16}{x^2-8x+16}$.	10) Find $\lim_{x\to 2}\frac{x^3-8}{x-2}$.
11) Find $\lim_{x\to 0}\frac{1}{x}$	12) Find $\lim_{x\to 1}\frac{1}{x-1}$
13) Find $\lim_{x\to 0}\frac{1}{x^2}$	14) Find $\lim_{x\to 0}\frac{-1}{x^2}$
15) Find $\lim_{x\to 0^+}\log(x)$	16) Find $\lim_{x\to 1}\frac{x}{x-1}$

Solutions:

1) Evaluate $\lim_{x\to 5}\frac{x^2-25}{x-5}$. - Factor the numerator as a difference of squares: $\lim_{x\to 5}\frac{(x+5)(x-5)}{x-5}=\lim_{x\to 5}(x+5)=5+5=10$	**2) Find the limit** $\lim_{x\to -3}\frac{x^2-9}{x+3}$. - Factor the numerator: $\lim_{x\to -3}\frac{(x-3)(x+3)}{x+3}=\lim_{x\to -3}(x-3)=-3-3=-6$
3) Compute $\lim_{x\to 2}\frac{x^2-4x+4}{x-2}$. - The numerator is a perfect square: $\lim_{x\to 2}\frac{(x-2)^2}{x-2}=\lim_{x\to 2}(x-2)=2-2=0$	**4) Determine** $\lim_{x\to 1}\frac{x^2-1}{x^2-x}$. - Factor both the numerator and the denominator: $\lim_{x\to 1}\frac{(x+1)(x-1)}{x(x-1)}=\lim_{x\to 1}\frac{x+1}{x}=\frac{2}{1}=2$
5) Evaluate $\lim_{x\to 0}\frac{x^2-4x}{x}$. - Factor x from the numerator: $\lim_{x\to 0}\frac{x(x-4)}{x}=\lim_{x\to 0}(x-4)=0-4=-4$	**6) Calculate** $\lim_{x\to 3}\frac{x^3-27}{x-3}$. - Use the difference of cubes to factor the numerator: $\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{x-3}$ $=\lim_{x\to 3}(x^2+3x+9)=3^2+3(3)+9=27$
7) Determine $\lim_{x\to -2}\frac{x^2-2x-8}{x+2}$. - Factor the numerator: $\lim_{x\to -2}\frac{(x-4)(x+2)}{x+2}=\lim_{x\to -2}(x-4)=-2-4=-6$	**8) Evaluate** $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$. - Use the difference of powers to factor: $\lim_{x\to 1}\frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)}=\lim_{x\to 1}\frac{(x+1)(x^2+1)}{(x^2+x+1)}$ $=\frac{2\times 2}{3}=\frac{4}{3}$
9) Calculate $\lim_{x\to 4}\frac{x^2-16}{x^2-8x+16}$. - Both the numerator and denominator can be factored: $\lim_{x\to 4}\frac{(x+4)(x-4)}{(x-4)(x-4)}=\lim_{x\to 4}\frac{x+4}{x-4}=\frac{8}{0}$ - However, since the denominator approaches 0 as x approaches 4, the limit **does not exist due to a division by zero**.	**10) Find** $\lim_{x\to 2}\frac{x^3-8}{x-2}$. - Factor the numerator using the difference of cubes: $\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}=\lim_{x\to 2}(x^2+2x+4)$ $=4+4+4=12$

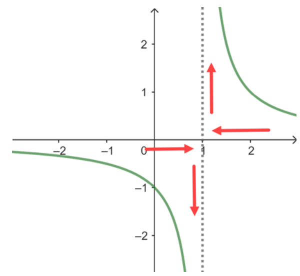
11) Find $\lim_{x\to 0}\frac{1}{x}$

- As x approaches 0, the value of $\frac{1}{x}$ grows without bound.
- Therefore, the limit **does not exist** because it approaches infinity.



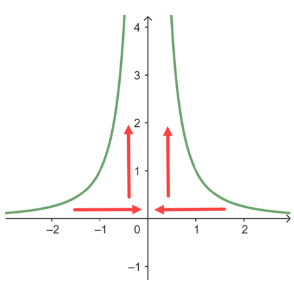
12) Find $\lim_{x\to 1}\frac{1}{x-1}$

- As x approaches 1, the denominator approaches 0, causing the fraction to approach infinity or negative infinity depending on the direction from which x approaches 1.
- Therefore, the limit **does not exist**.



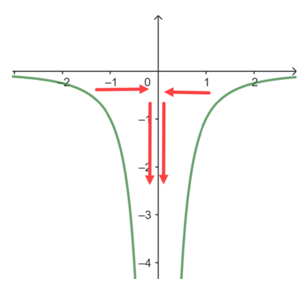
13) Find $\lim_{x\to 0}\frac{1}{x^2}$

- As x approaches 0, the value of $\frac{1}{x^2}$ increases without bound. So, the limit is infinity.
- As $x\to 0$ from the positive side ($x\to 0^+$), $\frac{1}{x^2}\to\infty$.
- As $x\to 0$ from the negative side ($x\to 0^-$), $\frac{1}{x^2}\to\infty$.
- $\lim_{x\to 0}\frac{1}{x^2}=\infty$



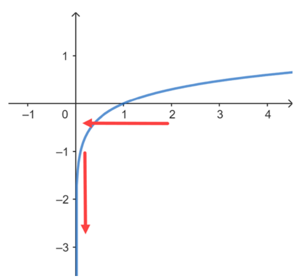
14) Find $\lim_{x\to 0}\frac{-1}{x^2}$

- As x approaches 0, $\frac{-1}{x^2}$ becomes increasingly negative without bound, so the limit is negative infinity ($-\infty$).
- As $x\to 0$ from the positive side (. $x\to 0^+$.), $\frac{-1}{x^2}\to-\infty$.
- As $x\to 0$ from the negative side ($x\to 0^-$), $\frac{-1}{x^2}\to-\infty$.



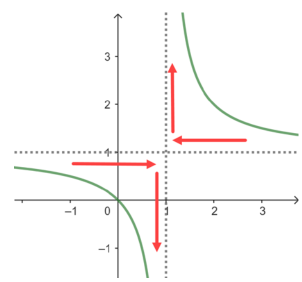
15) Find $\lim_{x\to 0^+} \log(x)$

- The natural logarithm of x as x approaches 0 from the right-hand side goes to negative infinity, since the logarithm of values between 0 and 1 is negative and decreases without bound as x approaches 0.
- As $x \to 0^+$ (approaching from the right), the logarithm function $\log(x)$ approaches $-\infty$.
- $\lim_{x\to 0^+} \log(x) = -\infty$



16) Find $\lim_{x\to 1} \frac{x}{x-1}$

- As x approaches 1, the denominator approaches 0, which makes the fraction undefined at x = 1.
- However, unlike the previous examples, the numerator also approaches 1, so we consider the limit from both sides. From both sides, as x gets closer to 1, the value of the fraction grows without bound. The limit **does not exist** and it can be said to approach infinity.
- As $x \to 1^+$ (approaching from the right):
  $\frac{x}{x-1} \to +\infty$
- As $x \to 1^-$ (approaching from the left):
  $\frac{x}{x-1} \to -\infty$



1-4. Limit Estimation Using Calculators (e.g., TI-84+)

Estimating limits using calculators like the TI-84+ can be particularly useful when dealing with complex functions where analytical solutions are difficult to derive or verify.

Calculators can provide numerical approximations to limits by evaluating the function at points increasingly close to the point of interest.

This method is not exact but offers practical insights into the behavior of the function near the target value.

Procedure

- Use the table feature of the calculator.
- Set the table to increment at smaller intervals around the limit point.
- Observe the function values as they converge to a point.

Examples:

1) Estimate $\lim_{x\to 2}\frac{x^2-4}{x-2}$ using a calculator's table feature.	2) Use a calculator to estimate $\lim_{x\to 0}\frac{1-\cos(x)}{x}$.
3) Find the limit: $\lim_{x\to 3}\frac{2x^2-5x-3}{x^2-9}$	4) Find the limit: $\lim_{x\to -1}\frac{x^3+1}{x+1}$

5) Estimate $\lim_{x\to 0^+} e^{-1/x}$ using your calculator.	6) Estimate $\lim_{x\to\infty}\frac{1}{x}$ using a calculator's table feature.
7) Use a calculator to estimate $\lim_{x\to 0}\frac{\sin(2x)}{x}$.	8) Estimate $\lim_{x\to 0}\frac{\sin(x)}{x^2}$ using a calculator.
9) Using a calculator, estimate $\lim_{x\to 0}\ln(1+x)$.	10) Estimate $\lim_{x\to 0}\frac{\arctan(x)}{x}$ with the help of a calculator.
11) Use a calculator to estimate $\lim_{x\to 0^+} x\ln(x)$.	12) Estimate $\lim_{x\to\infty}\frac{\ln(x)}{x}$ using a calculator.
13) Evaluate the limit: $\lim_{x\to 0^-}\frac{\lvert x\rvert}{x}$.	14) Evaluate the limit: $\lim_{x\to 0}\frac{\sin(4x)}{\sin(6x)}$.

Solutions:

1) Estimate $\lim_{x\to 2}\frac{x^2-4}{x-2}$ **using a calculator's table feature.**

- Set up the table feature to increment values of x around 2 (for example, 1.9, 1.99, 1.999 2.001, 2.01, 2.1) and observe the y values.
- The value should approach 4 as x **approaches 2**.

2) Use a calculator to estimate $\lim_{x\to 0}\frac{1-\cos(x)}{x}$ **.**

- Increment x values around 0 (e.g., 0.1, 0.01, 0.001, -0.001, -0.01, -0.1) and observe the y values.
- The value should **approach 0** as x approaches 0.

3) Find the limit: $\lim_{x\to 3}\frac{2x^2-5x-3}{x^2-9}$

- Simplified $\frac{2x^2-5x-3}{x^2-9}=\frac{(2x+1)(x-3)}{(x-3)(x+3)}=\frac{(2x+1)}{(x+3)}$
- Substitute x = 3 into the simplified expression: $\frac{2(3)+1}{3+3}=\frac{6+1}{6}=\frac{7}{6}$
- $\lim_{x\to 3}\frac{2x^2-5x-3}{x^2-9}=\frac{7}{6}$

4) Find the limit: $\lim_{x\to -1}\frac{x^3+1}{x+1}$

- Simplified: $\frac{x^3+1}{x+1}=\frac{(x+1)(x^2-x+1)}{x+1}=$ (x^2-x+1)
- Substitute x = -1 into the simplified expression: $(-1)^2-(-1)+1=1+1+1=3$
- $\lim_{x\to -1}\frac{x^3+1}{x+1}=3$

5) Estimate $\lim_{x\to 0^+} e^{-1/x}$ **using your calculator.**

- Use the table feature to approach 0 from the positive side (e.g., 0.1, 0.01, 0.001) and note the y values **converging to 0**.

6) Estimate $\lim_{x\to\infty}\frac{1}{x}$ **using a calculator's table feature.**

- Set up the table with increasingly large values of x (e.g., 10, 100, 1000) and observe the y values **approaching 0**.

7) Use a calculator to estimate $\lim_{x\to 0}\frac{\sin(2x)}{x}$ **.**

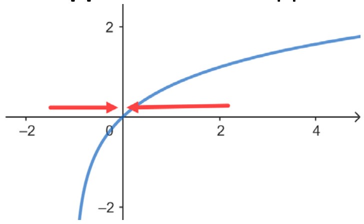
- Observe values of x close to 0 to see the y values converging, which should **approach 2** as x approaches 0 (e.g., 0.1, 0.01, 0.001).
- Or use: $\lim_{x\to 0}\frac{\sin(2x)}{x}=\lim_{x\to 0}2\cdot\frac{\sin(2x)}{2x}$ $=2\cdot\lim_{x\to 0}\frac{\sin(2x)}{2x}=2$ (knowing that $\lim_{u\to 0}\frac{\sin(u)}{u}=1$)

8) Estimate $\lim_{x\to 0}\frac{\sin(x)}{x^2}$ **using a calculator.**

- This function will **approach infinity (** $\pm\infty$ **)** as x approaches 0 (e.g., 0.1, 0.01, 0.001, -0.999, -0.99, -.9) from both sides, which you can observe by the y values increasing without bound.
- Or use: $\lim_{x\to 0}\frac{\sin(x)}{x\times x}=\lim_{x\to 0}\frac{1}{x}\frac{\sin(x)}{x}=\lim_{x\to 0}\frac{1}{x}=\pm\infty$

9) Using a calculator, estimate $\lim_{x\to 0} \ln(1+x)$.

A: Input $\ln(1+x)$ into the calculator and increment x around 0 (e.g., 0.1, 0.01, 0.001). The values should **approach 0** as x approaches 0.

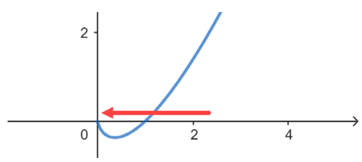


10) Estimate $\lim_{x\to 0} \frac{\arctan(x)}{x}$ with the help of a calculator.

A: Set the table to show values of x near 0 (e.g., 0.1, 0.01, 0.001) and observe that the y values **approach 1**.

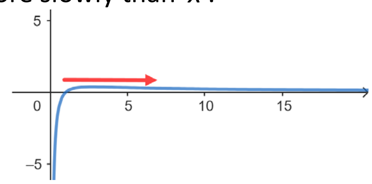
11) Use a calculator to estimate $\lim_{x\to 0^+} x\ln(x)$.

A: Increment x values close to 0 (e.g., 0.1, 0.01, 0.001) from the positive side and note that the y values will **approach 0.**



12) Estimate $\lim_{x\to\infty} \frac{\ln(x)}{x}$ using a calculator.

A: As x (e.g., 10, 100, 1000) becomes very large, observe the y values in the table. The y-value should **approach 0**, indicating that the function grows more slowly than x.



13) Evaluate the limit: $\lim_{x\to 0^-} \frac{|x|}{x}$.

To evaluate the limit $\lim_{x\to 0^-} \frac{|x|}{x}$, we consider the behavior of the function as x approaches 0from the negative side:

For $x < 0 : |x| = -x$

Therefore: $\frac{|x|}{x} = \frac{-x}{x} = -1$

So, $\lim_{x\to 0^-} \frac{|x|}{x} = -1$.

14) Evaluate the limit: $\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)}$.

To evaluate the limit $\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)}$, we can use the limit property $\lim_{x\to 0} \frac{\sin(kx)}{kx} = 1$ for any constant k.

First, we rewrite the limit:

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} = \lim_{x\to 0} \left(\frac{\sin(4x)}{4x} \cdot \frac{6x}{\sin(6x)} \cdot \frac{4}{6} \right)$$

Using the known limits:

$$\lim_{x\to 0} \frac{\sin(4x)}{4x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{6x}{\sin(6x)} = 1$$

$$\text{So: } \lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} = \lim_{x\to 0} \left(\frac{\sin(4x)}{4x} \cdot \frac{6x}{\sin(6x)} \cdot \frac{4}{6} \right)$$

$$= 1 \cdot 1 \cdot \frac{4}{6} = \frac{4}{6} = \frac{2}{3}$$

So, $\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} = \frac{2}{3}$.

1-5. Graphical Limit Estimation and Visualization

Graphical estimation of limits involves analyzing the behavior of a function as it approaches a certain point from the plot or graph of the function.

This method is particularly useful for visual learners and can provide intuitive insights into the behavior of functions near points of interest, including points of discontinuity or where the function does not have an explicit value.

Procedure

- Plot the graph of the function around the point of interest.
- Observe the behavior of the function as the input values get closer to the target point from both directions.

Examples:

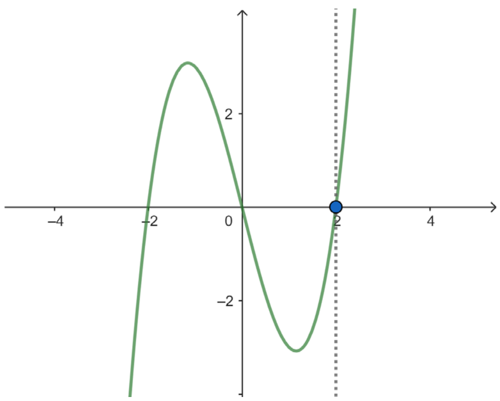
1) Graph the function $f(x)=x^3-4x$ and estimate $\lim_{x\to 2} f(x)$.	2) Use a graph to estimate the limit $\lim_{x\to 0}\frac{\sin(x)}{x}$.
3) Estimate $\lim_{x\to -1}\frac{1}{x^2+1}$ by graphing.	4) Graph $f(x)=\sqrt{x}$ to estimate $\lim_{x\to 4} f(x)$.

5) Use a graphing tool to estimate the limit $\lim_{x\to\infty}\frac{x}{x+1}$.	6) Graph $f(x)=e^{-x}$ and estimate $\lim_{x\to\infty} f(x)$.
7) Estimate $\lim_{x\to 0^-}\frac{1}{x}$ using a graph.	8) Graph the function $f(x)=\ln(x)$ to estimate $\lim_{x\to 0^+} f(x)$.
9) Use graphical analysis to estimate $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$.	10) Estimate $\lim_{x\to 0^+} x^x$ with the help of a graph.

Solutions:

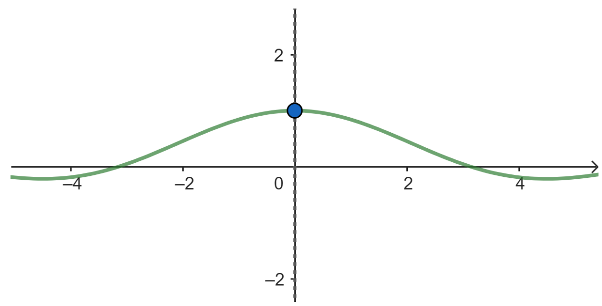
1) Graph the function $f(x) = x^3 - 4x$ and estimate $\lim_{x \to 2} f(x)$.

- By plotting the graph, you would observe that as x approaches 2, $f(x)$ approaches 0.
- Thus, $\lim_{x \to 2} f(x) = 0$.



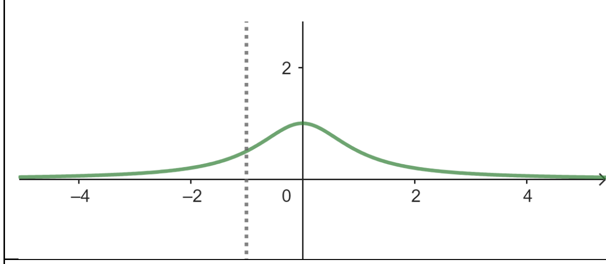
2) Use a graph to estimate the limit $\lim_{x \to 0} \frac{\sin(x)}{x}$.

- The graph of $\frac{\sin(x)}{x}$ will show that as x approaches 0 from both sides, $f(x)$ approaches 1.
- Hence, $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.



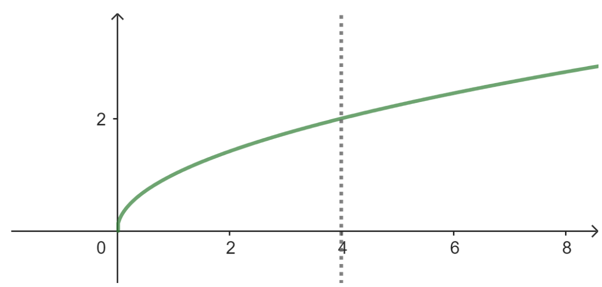
3) Estimate $\lim_{x \to -1} \frac{1}{x^2 + 1}$ by graphing.

- The graph of $\frac{1}{x^2 + 1}$ shows a horizontal asymptote as x approaches -1, and the y value is 0.5.
- So, $\lim_{x \to -1} \frac{1}{x^2 + 1} = 0.5$.



4) Graph $f(x) = \sqrt{x}$ to estimate $\lim_{x \to 4} f(x)$.

- The graph of $\sqrt{x}$ will show a smooth curve where as x approaches 4, $f(x)$ approaches 2. So, $\lim_{x \to 4} \sqrt{x} = 2$.



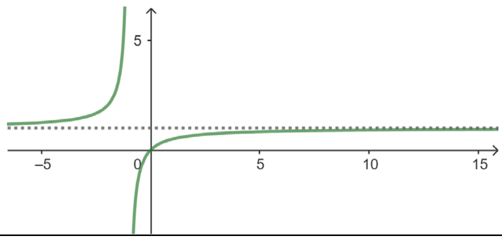
5) Use a graphing tool to estimate the limit $\lim_{x \to \infty} \frac{x}{x + 1}$.

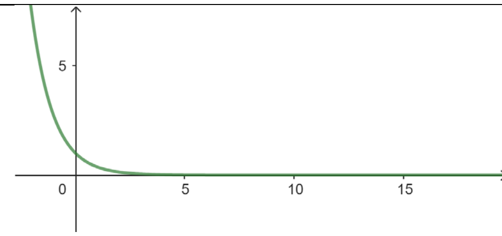
- The graph will show that as x increases without bound, $f(x)$ approaches the horizontal asymptote at $y = 1$.

6) Graph $f(x) = e^{-x}$ and estimate $\lim_{x \to \infty} f(x)$.

- The graph of e^{-x} will show a horizontal asymptote at $y = 0$, indicating that as x approaches infinity, $f(x)$ approaches 0.
- Therefore, $\lim_{x \to \infty} e^{-x} = 0$.

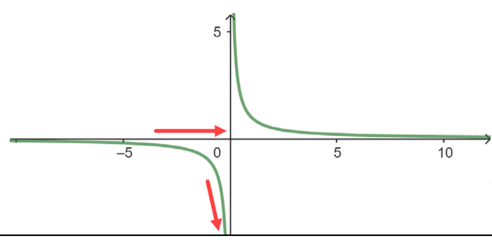
- Thus, $\lim_{x \to \infty} \frac{x}{x+1} = 1$.





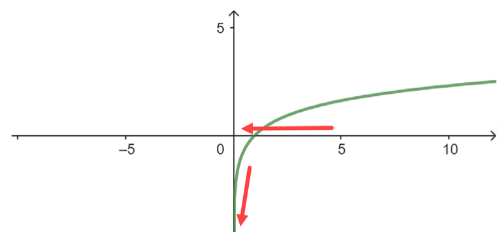
7) Estimate $\lim_{x \to 0^-} \frac{1}{x}$ using a graph.

- The graph of $\frac{1}{x}$ will show a vertical asymptote at $x = 0$.
- As x approaches 0 from the left, $f(x)$ decreases without bound, indicating a limit of $-\infty$.



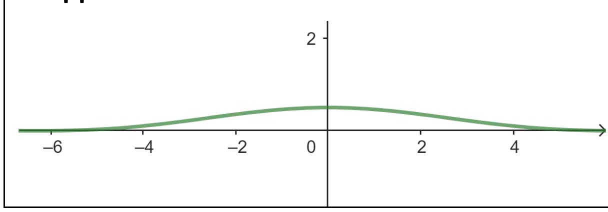
8) Graph the function $f(x) = \ln(x)$ to estimate $\lim_{x \to 0^+} f(x)$.

- The graph of $\ln(x)$ will show that as x approaches 0 from the right, $f(x)$ goes to $-\infty$.
- Thus, $\lim_{x \to 0^+} \ln(x) = -\infty$.



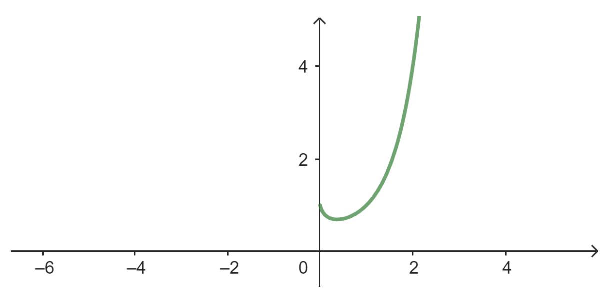
9) Use graphical analysis to estimate $\lim_{x \to 0} \frac{1-\cos(x)}{x^2}$.

- In this particular case, the function $\frac{1-\cos(x)}{x^2}$ does have a limit as x approaches 0.
- While we expect the function to have an indeterminate form of $\frac{0}{0}$ at $x = 0$, L'Hôpital's Rule or Taylor series expansion would show that the limit is actually 0.5.
- Therefore, a graph of this function would show that as x approaches 0, the **function approaches 0.5.**



10) Estimate $\lim_{x \to 0^+} x^x$ with the help of a graph.

- Plotting x^x and approaching 0 from the right, the graph shows $f(x)$ approaching 1.
- So, $\lim_{x \to 0^+} x^x = 1$.



1-6. Algebraic Manipulation and Properties of Limits

Algebraic manipulation in the context of limits involves rearranging and simplifying expressions to make limit calculations more straightforward.

Understanding the properties of limits is essential for effectively applying these techniques, as these properties allow the manipulation of limits in ways that are analogous to standard algebraic operations.

The properties of limits provide a set of rules that help in the computation of limits, ensuring that the limit of a combination of functions can often be determined from the limits of the individual functions, provided those limits exist.

These properties are crucial when dealing with sums, products, quotients, and compositions of functions.

- Sum/Difference Rule: $\lim_{x\to a}[f(x)\pm g(x)]=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)$
- Product Rule: $\lim_{x\to a}[f(x)\cdot g(x)]=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$
- Quotient Rule: $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$ if $\lim_{x\to a}g(x)\neq 0$
- Constant Multiple Rule: $\lim_{x\to a}c\cdot f(x)=c\cdot\lim_{x\to a}f(x)$

Examples:

1) Evaluate the limit $\lim_{x\to 2}(3x+2)$.	2) Find the limit $\lim_{x\to -1}(4-5x)$.

Examples:

1) Taylor Polynomial of e^x **Centered at** $a = 0$ **(Maclaurin Series)**

- Find the Maclaurin polynomial of degree 4 for $f(x) = e^x$.

2) Taylor Polynomial of $\sin(x)$ **centered at** $a = 0$ **(Maclaurin Series)**

- Find the Maclaurin polynomial of degree 5 for $f(x) = \sin(x)$.

3) Taylor Polynomial of $\ln(1+x)$ **Centered at** $a = 0$ **(Maclaurin Series)**

- Find the Maclaurin polynomial of degree 3 for $f(x) = \ln(1+x)$.

4) Taylor Polynomial of $\cos(x)$ **Centered at** $a = \pi / 4$

- Find the Taylor polynomial of degree 2 for $f(x) = \cos(x)$ centered at $a = \pi / 4$.

Solutions:

1) Taylor Polynomial of e^x Centered at $a=0$ (Maclaurin Series)

- **Find the Maclaurin polynomial of degree 4 for $f(x)=e^x$.**

- For $f(x)=e^x$, all derivatives are $f^{(k)}(x)=e^x$, and evaluated at $x=0$, we have $f^{(k)}(0)=1$.

- The Maclaurin polynomial of degree 4 is:

$$P_4(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}$$

$$P_4(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$$

2) Taylor Polynomial of $\sin(x)$ centered at $a=0$ (Maclaurin Series)

- **Find the Maclaurin polynomial of degree 5 for $f(x)=\sin(x)$.**

- For $f(x)=\sin(x)$, the derivatives cycle as follows:

$$f'(x)=\cos(x),\quad f''(x)=-\sin(x),\quad f'''(x)=-\cos(x),\quad f^{(4)}(x)=\sin(x),\quad \text{and so on}$$

- Evaluating at $x=0$:

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=-1,\quad f^{(4)}(0)=0,\quad f^{(5)}(0)=1$$

- The Maclaurin polynomial of degree 5 is: $P_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}$

$$P_5(x)=x-\frac{x^3}{6}+\frac{x^5}{120}$$

3) Taylor Polynomial of $\ln(1+x)$ Centered at $a=0$ (Maclaurin Series)

- **Find the Maclaurin polynomial of degree 3 for $f(x)=\ln(1+x)$.**

- For $f(x)=\ln(1+x)$, the derivatives are:

$$f'(x)=\frac{1}{1+x},\quad f''(x)=-\frac{1}{(1+x)^2},\quad f'''(x)=\frac{2}{(1+x)^3}$$

- Evaluating at $x=0$:

$$f(0)=\ln(1)=0,\quad f'(0)=1,\quad f''(0)=-1,\quad f'''(0)=2$$

- The Maclaurin polynomial of degree 3 is:

$$P_3(x)=0+x-\frac{x^2}{2!}+\frac{2x^3}{3!}$$

$$P_3(x)=x-\frac{x^2}{2}+\frac{x^3}{3}$$

4) Taylor Polynomial of $\cos(x)$ Centered at $a=\pi/4$

- **Find the Taylor polynomial of degree 2 for $f(x)=\cos(x)$ centered at $a=\pi/4$.**

- For $f(x)=\cos(x)$, the derivatives are:

$$f'(x)=-\sin(x),\quad f''(x)=-\cos(x),\quad f'''(x)=\sin(x)$$

- Evaluating at $x=\pi/4$:

$$f(\pi/4)=\cos(\pi/4)=\frac{\sqrt{2}}{2},\quad f'(\pi/4)=-\sin(\pi/4)=-\frac{\sqrt{2}}{2},\quad f''(\pi/4)=-\cos(\pi/4)=-\frac{\sqrt{2}}{2}$$

- The Taylor polynomial of degree 2 is:

$$P_2(x)=\cos(\pi/4)-\sin(\pi/4)(x-\pi/4)-\frac{\cos(\pi/4)}{2!}(x-\pi/4)^2$$

$$P_2(x)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}(x-\pi/4)-\frac{\sqrt{2}}{4}(x-\pi/4)^2$$

11-12. Lagrange Error Bound

The **Lagrange Error Bound** provides a way to estimate the error when using a Taylor polynomial to approximate a function. This error bound is particularly useful for understanding how closely the Taylor polynomial approximates the actual function.

For a function f that is approximated by its Taylor polynomial $P_n(x)$ of degree n centered at a , the Lagrange Error Bound gives an estimate for the error $R_n(x)$ at a point x within the interval of approximation. The error bound is given by:

$$|R_n(x)| = \left|f(x) - P_n(x)\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

where M is an upper bound for the absolute value of the $(n+1)$-th derivative of f on the interval containing a and x .

Examples:

1) Estimating the Error for e^x

- Approximate e^x using the second-degree Maclaurin polynomial $P_2(x) = 1 + x + \frac{x^2}{2}$ and find the error bound for $x = 0.1$.

2) Estimating the Error for $\sin(x)$

- Approximate $\sin(x)$ using the third-degree Maclaurin polynomial $P_3(x) = x - \frac{x^3}{6}$ and find the error bound for $x = 0.5$.

3) Estimating the Error for $\ln(1+x)$

- Approximate $\ln(1+x)$ using the second-degree Taylor polynomial centered at $a = 0$, $P_2(x) = x - \frac{x^2}{2}$, and find the error bound for $x = 0.2$.

Solutions:

1) Estimating the Error for e^x

- **Approximate e^x using the second-degree Maclaurin polynomial $P_2(x)=1+x+\frac{x^2}{2}$ and find the error bound for $x=0.1$.**

- The Maclaurin polynomial of degree 2 for e^x is: $P_2(x)=1+x+\frac{x^2}{2}$
- The $(n+1)$-th derivative for e^x is also e^x. Since e^x is increasing, the maximum value of e^x on the interval [0, 0.1] occurs at $x=0.1$: $M=e^{0.1}\approx 1.105$
- Apply the Lagrange Error Bound:

$$|R_2(0.1)|\leq \frac{M}{3!}|0.1-0|^3=\frac{1.105}{6}(0.1)^3=\frac{1.105}{6000}\approx 0.000184$$

- So, the error bound for the approximation at $x=0.1$ is approximately 0.000184.

2) Estimating the Error for $\sin(x)$

- **Approximate $\sin(x)$ using the third-degree Maclaurin polynomial $P_3(x)=x-\frac{x^3}{6}$ and find the error bound for $x=0.5$.**

- The Maclaurin polynomial of degree 3 for $\sin(x)$ is: $P_3(x)=x-\frac{x^3}{6}$
- The $(n+1)$-th derivative for $\sin(x)$ is $\cos(x)$ or $-\cos(x)$. The maximum value of $|\cos(x)|$ on the interval $[-0.5, 0.5]$ is 1.
- Apply the Lagrange Error Bound:

$$|R_3(0.5)|\leq \frac{M}{4!}|0.5-0|^4=\frac{1}{24}(0.5)^4=\frac{1}{24}\cdot\frac{1}{16}=\frac{1}{384}\approx 0.0026$$

- So, the error bound for the approximation at $x=0.5$ is approximately 0.0026.

3) Estimating the Error for $\ln(1+x)$

- **Approximate $\ln(1+x)$ using the second-degree Taylor polynomial centered at $a=0$, $P_2(x)=x-\dfrac{x^2}{2}$, and find the error bound for $x=0.2$.**

- The Taylor polynomial of degree 2 for $\ln(1+x)$ is: $P_2(x)=x-\dfrac{x^2}{2}$
- The $(n+1)$-th derivative for $\ln(1+x)$ is $\dfrac{(-1)^n}{(1+x)^n}$.
- The maximum value of $\left|\dfrac{(-1)^3}{(1+x)^3}\right|=\dfrac{1}{(1+x)^3}$ on the interval [0, 0.2] is at $x=0$: $M=\dfrac{1}{1^3}=1$
- Apply the Lagrange Error Bound:

$$|R_2(0.2)|\le\frac{M}{3!}|0.2-0|^3=\frac{1}{6}(0.2)^3=\frac{1}{6}\cdot 0.008=\frac{0.008}{6}\approx 0.00133$$

- So, the error bound for the approximation at $x=0.2$ is approximately 0.00133.

11-13. Understanding Euler's Method (Review from 9-3)

Euler's Method is a numerical technique used to approximate solutions to ordinary differential equations (ODEs). Given an initial value problem of the form:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Euler's Method approximates the solution by iterating over small steps, using the slope at each point to estimate the next point.

Steps:

1. Starting Point: Begin with the initial condition (x_0, y_0).
2. Step Size: Choose a step size h .
3. Iteration: Use the following formula to find successive points:

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$$x_{n+1} = x_n + h$$

This process is repeated for the desired number of steps or until a specific value of x is reached.

Examples:

1) Approximating y for a Simple Differential Equation

- Use Euler's Method to approximate the solution to $\frac{dy}{dx} = x + y$ with $y(0) = 1$ at $x = 0.2$ using a step size of $h = 0.1$.

2) Approximating y for a Differential Equation with Nonlinear Terms

- Use Euler's Method to approximate the solution to $\frac{dy}{dx} = y - x^2 + 1$ with $y(0) = 0.5$ at $x = 0.2$ using a step size of $h = 0.1$.

Solutions:

1) Approximating y for a Simple Differential Equation

- **Use Euler's Method to approximate the solution to** $\frac{dy}{dx} = x + y$ **with** $y(0) = 1$ **at** $x = 0.2$ **using a step size of** $h = 0.1$ **.**

- Initial condition: $x_0 = 0$, $y_0 = 1$, Step size: $h = 0.1$
- Iteration 1:
  - $f(x_0, y_0) = f(0,1) = 0 + 1 = 1$
  - $y_1 = y_0 + h \cdot f(x_0, y_0) = 1 + 0.1 \cdot 1 = 1.1$
  - $x_1 = x_0 + h = 0 + 0.1 = 0.1$
- Iteration 2:
  - $f(x_1, y_1) = f(0.1, 1.1) = 0.1 + 1.1 = 1.2$
  - $y_2 = y_1 + h \cdot f(x_1, y_1) = 1.1 + 0.1 \cdot 1.2 = 1.22$
  - $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
- Therefore, the approximate value of y at $x = 0.2$ is $y \approx 1.22$.

2) Approximating y for a Differential Equation with Nonlinear Terms

- **Use Euler's Method to approximate the solution to** $\frac{dy}{dx} = y - x^2 + 1$ **with** $y(0) = 0.5$ **at** $x = 0.2$ **using a step size of** $h = 0.1$ **.**

- Initial condition: $x_0 = 0$, $y_0 = 0.5$, Step size: $h = 0.1$
- Iteration 1:
  - $f(x_0, y_0) = f(0, 0.5) = 0.5 - 0^2 + 1 = 1.5$
  - $y_1 = y_0 + h \cdot f(x_0, y_0) = 0.5 + 0.1 \cdot 1.5 = 0.65$
  - $x_1 = x_0 + h = 0 + 0.1 = 0.1$
- Iteration 2:
  - $f(x_1, y_1) = f(0.1, 0.65) = 0.65 - (0.1)^2 + 1 = 1.65 - 0.01 = 1.64$
  - $y_2 = y_1 + h \cdot f(x_1, y_1) = 0.65 + 0.1 \cdot 1.64 = 0.814$
  - $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$
- Therefore, the approximate value of y at $x = 0.2$ is $y \approx 0.814$.

11-14. Finding Taylor or Maclaurin Series for a Function

The **Taylor series** of a function provides a polynomial approximation of the function around a point a. When the point a is zero, the series is known as the Maclaurin series. These series are powerful tools for approximating functions using an infinite sum of terms derived from the function's derivatives at a single point.

Formulas:

1. Taylor Series: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$ where $f^{(n)}(a)$ is the n-th derivative of f evaluated at $x=a$

2. Maclaurin Series (special case of the Taylor series at $a=0$): $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

Examples:

1) Maclaurin Series for e^x

- Find the Maclaurin series for $f(x)=e^x$.

2) Taylor Series for $\sin(x)$ **centered at** $a=\pi/4$

- Find the Taylor series for $f(x)=\sin(x)$ centered at $a=\pi/4$.

3) Maclaurin Series for $\cos(x)$

- Find the Maclaurin series for $f(x)=\cos(x)$.

Solutions:

1) Maclaurin Series for e^x

- **Find the Maclaurin series for $f(x)=e^x$.**

- The function e^x and all its derivatives are e^x. At $x=0$, we have $f^{(n)}(0)=1$ for all n.
- The Maclaurin series is: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

2) Taylor Series for $\sin(x)$ centered at $a=\pi/4$

- **Find the Taylor series for $f(x)=\sin(x)$ centered at $a=\pi/4$.**

- The derivatives of $\sin(x)$ cycle as follows:

$$f(x)=\sin(x),\ f'(x)=\cos(x),\ f''(x)=-\sin(x),\ f'''(x)=-\cos(x)$$

- Evaluating at $x=\pi/4$:

$$f(\pi/4)=\sin(\pi/4)=\frac{\sqrt{2}}{2},\ f'(\pi/4)=\cos(\pi/4)=\frac{\sqrt{2}}{2}$$
$$f''(\pi/4)=-\sin(\pi/4)=-\frac{\sqrt{2}}{2},\ f'''(\pi/4)=-\cos(\pi/4)=-\frac{\sqrt{2}}{2}$$

- The Taylor series is:

$$\sin(x)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\pi/4)-\frac{\sqrt{2}}{4!}(x-\pi/4)^2-\frac{\sqrt{2}}{3!}(x-\pi/4)^3+\cdots$$

3) Maclaurin Series for $\cos(x)$

- **Find the Maclaurin series for $f(x)=\cos(x)$.**

- The derivatives of $\cos(x)$ cycle as follows:

$$f(x)=\cos(x),\quad f'(x)=-\sin(x),\quad f''(x)=-\cos(x),\quad f'''(x)=\sin(x)$$

- Evaluating at $x=0$:

$$f(0)=1,\quad f'(0)=0,\quad f''(0)=-1,\quad f'''(0)=0$$

- The Maclaurin series is:

$$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots$$

11-15. Representing Functions as Power Series

A **power series** is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where c_n are the coefficients, a is the center of the series, and x is the variable. Power series can represent functions within their radius of convergence. Many functions can be expressed as power series, allowing for powerful techniques in analysis and approximation.

Theorems and Tests:

1. **Radius of Convergence**: The radius of convergence R of a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is found using:

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n} \quad \text{or the Ratio Test: } R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

2. **Interval of Convergence:** The interval of convergence is the set of x $-values for which the series converges. This interval is (a-R, a+R) , and endpoints must be checked separately.

Examples:

1) Power Series Representation of $\frac{1}{1-x}$

- Find the power series representation of $f(x) = \frac{1}{1-x}$ centered at $a = 0$.

2) Power Series Representation of e^x

- Find the power series representation of $f(x) = e^x$ centered at $a = 0$.

3) Power Series Representation of $\sin(x)$

- Find the power series representation of $f(x) = \sin(x)$ centered at $a = 0$.

Solutions:

1) Power Series Representation of $\frac{1}{1-x}$

- **Find the power series representation of $f(x)=\frac{1}{1-x}$ centered at $a=0$.**

- The function $\frac{1}{1-x}$ can be expressed as a geometric series for $|x|<1$:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

- The radius of convergence R is 1, and the interval of convergence is $(-1,1)$.

2) Power Series Representation of e^x

- **Find the power series representation of $f(x)=e^x$ centered at $a=0$.**

- The function e^x and all its derivatives are e^x. The Maclaurin series is:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

- The radius of convergence R is infinite, and the interval of convergence is $(-\infty,\infty)$.

3) Power Series Representation of $\sin(x)$

- **Find the power series representation of $f(x)=\sin(x)$ centered at $a=0$.**

- The Maclaurin series for $\sin(x)$ is:

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

- The radius of convergence R is infinite, and the interval of convergence is $(-\infty,\infty)$.

eSpyMath: AP Calculus AB/BC Workbook

End of the Workbook

Congratulations!

I'm glad I could assist you in gaining math knowledge for Calculus. If you have any more questions or need further assistance in the future, feel free to ask.

Good luck with your studies!

About Authors

Seonwan Myung, Ph.D.

Seonwan Myung holds a Ph.D. and master's degrees in industrial engineering and has a keen interest in making math easy to learn and study. With extensive experience in math teaching and tutoring, he specializes in designing enterprise applications, particularly in software design.

Jaden Myung

Jaden Myung is a high school with impressive achievements in mathematics. He completed an AP Precalculus course in 2023 as a freshman, excelled in Pre-Algebra and Algebra, received Academic Excellence awards in math, and won the Pythagoras Award in 2022 and 2023. These accomplishments showcase his strong mathematical skills and dedication to academic success.

eSpyMath Books

eSpyMath Pre-Algebra Workbook

eSpyMath Algebra 1 Workbook

eSpyMath Algebra 2 Workbook

eSpyMath AP Pre-Calculus Workbook

eSpyMath AP Calculus AB/BC Workbook

eSpyMath AP Calculus AB/BC Textbook